Sloshing theory

Peng

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1 Governing equations

1.1 NS eqns in ICS

For incompressible fluids, $\rho = \text{const}$

1. Generalized Gauss theorem:

$$\begin{cases} \int_Q \nabla x dQ = \int_{S_Q} \vec{n} x dS \\ \int_Q \nabla \cdot \vec{x} dQ = \int_{S_Q} \vec{n} \cdot \vec{x} dS \\ \int_Q \nabla \times \vec{x} dQ = \int_{S_Q} \vec{n} \times \vec{x} dS \end{cases} \tag{1}$$

2. Continuity equation for incompressible fluids in ICS (Eulerian description for a fixed point (x', y', z'), where x', y', z', t are independent with each other)

$$\nabla \cdot \boldsymbol{v}'(x', y', z', t) = 0 \tag{2}$$

Considering a small fluid domain ΔQ not change with time

$$\text{Mass flux} = \rho \int_{S_{\Delta Q}} \boldsymbol{v}' \cdot \vec{n} dS = \rho \int_{\Delta Q} \nabla \cdot \boldsymbol{v}' dQ = 0 \Longrightarrow \nabla \cdot \boldsymbol{v}' = 0$$

3. Momentum equation: surface forces(due to pressure + viscous stresses) + body forces = momentum flux + time rate of change of the momentum inside the fluid domain ΔQ

$$\begin{cases} \rho \frac{D\boldsymbol{v}'(t)}{Dt} = \rho \left[\frac{\partial \boldsymbol{v}'(x', y', z', t)}{\partial t} + \boldsymbol{v}'(x', y', z', t) \cdot \nabla \boldsymbol{v}'(x', y', z', t) \right] \\ = \rho \left[u' \frac{\partial \boldsymbol{v}'(x', y', z', t)}{\partial x'} + v' \frac{\partial \boldsymbol{v}'(x', y', z', t)}{\partial y'} + w' \frac{\partial \boldsymbol{v}'(x', y', z', t)}{\partial z'} \right] \\ = -\nabla p + \rho \vec{g} + [\mu \nabla^2 \boldsymbol{v}'(x', y', z', t) = \nabla \cdot (\tau_{ij} \vec{e}_i \vec{e}_j)] \\ \mu = 0 \xRightarrow{\text{Eulerian eqution}} \rho \frac{D\boldsymbol{v}'(t)}{Dt} = -\nabla p + \rho \vec{g} \end{cases} \tag{3}$$

Material derivative

$$\underbrace{\frac{D\vec{A}(t)}{Dt}}_{\text{Lagrangian description}} = \underbrace{\frac{\partial \vec{A}(x,y,z,t)}{\partial t} + \vec{v}(x,y,z,t)\cdot\nabla\vec{A}(x,y,z,t)}_{\text{Eulerian description}}$$
$$= \frac{\partial \vec{A}(x,y,z,t)}{\partial t} + u\frac{\partial \vec{A}(x,y,z,t)}{\partial x} + v\frac{\partial \vec{A}(x,y,z,t)}{\partial y} + w\frac{\partial \vec{A}(x,y,z,t)}{\partial z} \quad (4)$$

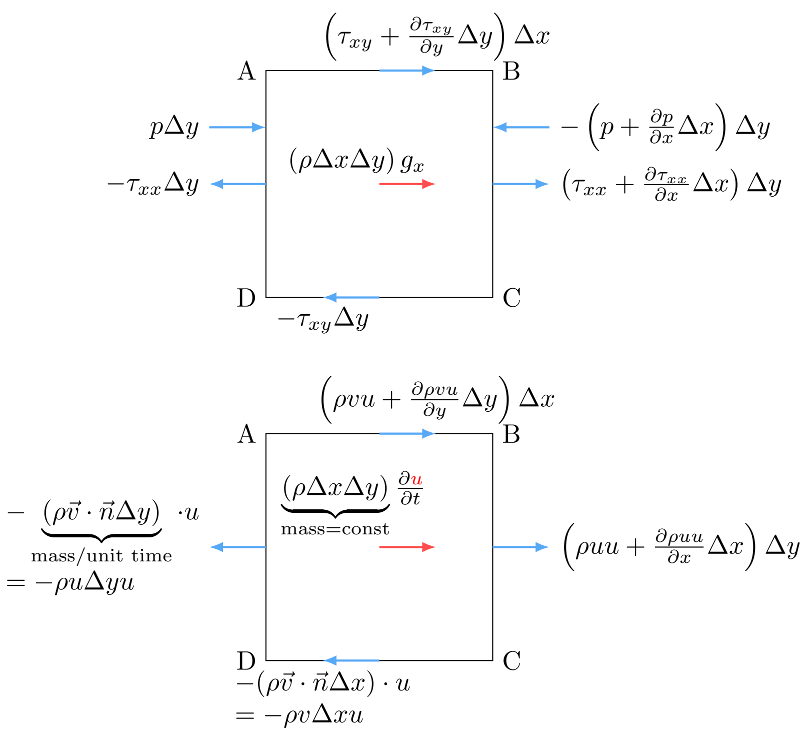

Figure 1: Surface/body forces and momentum flux/inner momentum in x direction at the boundary/in the domain of the fluid domain ΔQ.

$$\vec{F} = \underbrace{\text{pressure}\cdot S_Q + \text{viscous stresses}\cdot S_Q}_{\text{surface forces}} + \underbrace{\rho V_Q\cdot\vec{g}}_{\text{body forces}}$$
$$= \text{boundary momentum flux} + \frac{\partial}{\partial t}(\text{inner momentum})$$

(a) Forces in x direction:

i. AD+BC:

$$\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}\Delta x - \tau_{xx}\right)\Delta y - \frac{\partial p}{\partial x}\Delta x\Delta y = \frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right)\Delta x\Delta y - \frac{\partial p}{\partial x}\Delta x\Delta y$$

ii. AB+CD:

$$\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\Delta y - \tau_{xy}\right)\Delta x = \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]\Delta x\Delta y$$

iii. Sum:

$$\Delta x\Delta y\left[\mu\left(2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\partial y}\right) - \frac{\partial p}{\partial x} + \rho g_1\right]$$
$$= \Delta x\Delta y\left[\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial x}\left(\underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}}_{=0}\right)\right) - \frac{\partial p}{\partial x} + \rho g_1\right]$$

(b) Time rate of change of momentum in x direction:

i. Momentum flux through boundaries:

$$\rho\Delta x\Delta y\left(\frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y}\right)$$

ii. Time rate of change of momentum inside the fluid domain ΔQ:

$$\rho\Delta x\Delta y\frac{\partial u}{\partial t}$$

iii. Sum:

$$\rho\Delta x\Delta y\left(\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y}\right) = \rho\Delta x\Delta y\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + u\left(\underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}}_{=0}\right)\right]$$

(c) $\vec{F} = \frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t} \implies \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g_1$

(d) $\vec{v}\cdot\nabla\vec{v}$ and $\mu\nabla^2\vec{v} = \nabla\cdot\tau_{ij}\vec{e_i}\vec{e_j}$:

i.

$$\boldsymbol{a}\cdot\boldsymbol{A} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\boldsymbol{A}\cdot\boldsymbol{a} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

ii. See 1 and 2, which one is correct?

$$
\begin{aligned}
\vec{v}\cdot\nabla\vec{v} &= \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \\
&= \begin{bmatrix} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{bmatrix}^T \\
&= u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + z\frac{\partial \vec{v}}{\partial z}
\end{aligned}
$$

iii.

$$
\begin{aligned}
\mu\nabla^2\vec{v} = \mu\nabla\cdot\nabla\vec{v} &= \mu \begin{bmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \\
&= \mu \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{bmatrix}^T \\
&= \mu\left(\frac{\partial^2 \vec{v}}{\partial x^2} + \frac{\partial^2 \vec{v}}{\partial y^2} + \frac{\partial^2 \vec{v}}{\partial z^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot\tau_{ij}\vec{e_i}\vec{e_j} &= \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \end{bmatrix}^T \\
&= \mu \begin{bmatrix} 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2\frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + 2\frac{\partial^2 w}{\partial z^2} \end{bmatrix}^T \\
&= \mu \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \end{bmatrix}^T \\
&= \mu \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{bmatrix}^T \\
&= \mu\nabla^2\vec{v}
\end{aligned}
$$

1.2 Governing equations in ICS for potential flows

1. $\varpi = \nabla \times \vec{v} = \nabla \times \nabla\Phi = 0$

$$\vec{\varpi} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$
$$= \vec{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \vec{j}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

2. $\nabla \cdot \vec{v} = \nabla \cdot \nabla\Phi = \nabla^2\Phi = 0$

Laplace equation for potential flows in ICS (Eulerian description for a fixed point (x',y',z'))

$$\nabla^2\Phi(x', y', z', t) = \frac{\partial^2\Phi}{\partial x'^2} + \frac{\partial^2\Phi}{\partial y'^2} + \frac{\partial^2\Phi}{\partial z'^2} = 0 \tag{5}$$

3. Bernoulli's equation for potential flows in ICS

$$p + \rho\left[\frac{\partial\Phi(x', y', z', t)}{\partial t} + \frac{(\nabla\Phi(x', y', z', t))^2}{2} + U_g\,(= -\vec{g}\cdot\vec{r}(x', y', z'))\right] = C(t) \tag{6}$$

Eulerian equation $\Longrightarrow$ Bernoulli's equation, see 1 and 2.

$$\rho\frac{D\vec{v}}{Dt} = -\nabla p + \rho\vec{g}$$
$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla p - \rho\nabla U_g$$
$$\rho\left[\frac{\partial\nabla\Phi}{\partial t} + \nabla\frac{(\nabla\Phi)^2}{2}\right] = -\nabla p - \rho\nabla U_g$$
$$\nabla\left[\rho\frac{\partial\Phi}{\partial t} + \rho\frac{(\nabla\Phi)^2}{2}\right] = \nabla(-p - \rho U_g)$$
$$\nabla\left[p + \rho\left(\frac{\partial\Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} + U_g\right)\right] = 0$$

(a) $(\vec{v}\cdot\nabla)\vec{v} = \nabla\left(\frac{\vec{v}\cdot\vec{v}}{2}\right) - \vec{v}\times\underbrace{(\nabla\times\vec{v})}_{=\nabla\times\nabla\Phi=0} = \nabla\frac{(\nabla\Phi)^2}{2}$

1.3 Global conservation laws

Reynolds transport theorem:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\int_{Q(t)} X(x,y,z,t)dQ(t) = \int_{Q(t)} \frac{\partial X(x,y,z,t)}{\partial t}dQ(t) + \int_{S_Q(t)} X(x,y,z,t)U_{sn}dS(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\int_{Q(t)} \vec{X}(x,y,z,t)dQ(t) = \int_{Q(t)} \frac{\partial \vec{X}(x,y,z,t)}{\partial t}dQ(t) + \int_{S_Q(t)} \vec{X}(x,y,z,t)U_{sn}dS(t) \end{cases} \tag{7}$$

1.3.1 Conservation of fluid momentum

Incompressible fluid, ICS, $\vec{M}(t) = (M_1(t), M_2(t), M_3(t)) = \int_{Q(t)} \rho\vec{v}dQ(t)$

$$\begin{aligned} \frac{\mathrm{d}\vec{M}(t)}{\mathrm{d}t} &= \rho\int_{Q(t)} \frac{\partial \vec{v}}{\partial t}dQ(t) + \rho\int_{S_{Q(t)}} \vec{v}U_{sn}dS(t) \\ &= \rho\int_{Q(t)} \left[-\frac{1}{\rho}\nabla(p+\rho gz) + \frac{1}{\rho}\nabla\cdot(\tau_{ij}\vec{e}_i\vec{e}_j) - \nabla\cdot(\vec{v}\vec{v})\right]dQ(t) + \rho\int_{S_{Q(t)}} \vec{v}U_{sn}dS(t) \\ &= -\int_{S_Q(t)} \vec{n}pdS(t) - \int_{S_Q(t)} \vec{n}\rho gzdS(t) - \rho\int_{S_Q(t)} \vec{n}\cdot(\vec{v}\vec{v})dS(t) + \rho\int_{S_{Q(t)}} \vec{v}U_{sn}dS(t) \\ &\quad + \int_{Q(t)} \nabla\cdot(\tau_{ij}\vec{e}_i\vec{e}_j)dQ(t) \\ &= -\int_{S_Q(t)} \vec{n}pdS(t) - \int_{S_Q(t)} \vec{n}\rho gzdS(t) - \rho\int_{S_Q(t)} (\vec{v}u_n - U_{sn})dS(t) \\ &\quad + \int_{Q(t)} \nabla\cdot(\tau_{ij}\vec{e}_i\vec{e}_j)dQ(t) \end{aligned}$$

1. $\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla(p+\rho gz) + \nabla\cdot(\tau_{ij}\vec{e}_i\vec{e}_j)$

$$\rho\frac{\partial \vec{v}}{\partial t} = -\nabla(p+\rho gz) + \nabla\cdot(\tau_{ij}\vec{e}_i\vec{e}_j) - \rho\nabla\cdot(\vec{v}\vec{v})$$

2. $\nabla\cdot(\vec{v}\vec{v}) = \vec{v}(\nabla\cdot\vec{v} = 0) + \vec{v}\cdot\nabla\vec{v}$

$$\begin{aligned} \nabla\cdot(\vec{v}\vec{v}) &= \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{bmatrix} \\ &= \begin{bmatrix} u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\right) + \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \\ v\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\right) + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \\ w\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\right) + \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) \end{bmatrix}^T \end{aligned}$$

3. $\vec{g} = -\nabla(U_g = -\vec{g}\cdot\vec{r}) = \nabla((0,0,-g)\cdot(x,y,z)) = -\nabla gz$

4. $\vec{n} \cdot (\vec{v}\vec{v}) = u_n \vec{v}$

$$\begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{bmatrix} = \begin{bmatrix} u(un_1 + vn_2 + wn_3) \\ v(un_1 + vn_2 + wn_3) \\ w(un_1 + vn_2 + wn_3) \end{bmatrix}^T$$

1.3.2 Conservation of kinetic and potential fluid energy

$$E(t) = \rho \int_{Q(t)} \left(\frac{1}{2} \vec{v} \cdot \vec{v} + gz \right) dQ(t)$$

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = \rho \int_{Q(t)} \vec{v} \cdot \frac{\partial \vec{v}}{\partial t} dQ(t) + \rho \int_{S_Q(t)} \left(\frac{\vec{v} \cdot \vec{v}}{2} + gz \right) U_{sn} dS(t)$$

$$= \int_{Q(t)} \left\{ -\nabla \cdot \left[\rho \left(\frac{\vec{v} \cdot \vec{v}}{2} + \frac{p}{\rho} + gz \right) \vec{v} - \vec{v} \cdot \vec{\tau} \right] - \tau_{ij} \frac{\partial u_i}{\partial x_j} \right\} dQ(t) + \rho \int_{S_Q(t)} \left(\frac{\vec{v} \cdot \vec{v}}{2} + gz \right) U_{sn} dS(t)$$

$$= -\rho \int_{S_Q(t)} \left(\frac{\vec{v} \cdot \vec{v}}{2} + gz \right) (u_n - U_{sn}) dS - \int_{S_Q(t)} u_n p dS(t) + \int_{S_Q(t)} \vec{n} \cdot (\vec{v} \cdot \vec{\tau}) dS(t) - \int_{Q(t)} \tau_{ij} \frac{\partial u_i}{\partial x_j} dQ(t)$$

Conservation of energy for liquid motion inside a 2D tank :

On wetted surface: $u_n = U_{sn} = U_n, u_t = U_{st} = U_t$ (slip bounday condition)

For potential flow, no-requirement for a no-slip boundary condition

$$\vec{v} \cdot \vec{\tau} = (u_t \vec{t} + u_n \vec{n}) \cdot (\tau_{tt} \vec{t}\vec{t} + \tau_{tn} \vec{t}\vec{n} + \tau_{nt} \vec{n}\vec{t} + \tau_{nn} \vec{n}\vec{n})$$
$$= u_t \tau_{tt} \vec{t} + u_t \tau_{tn} \vec{n} + u_n \tau_{nt} \vec{t} + u_n \tau_{nn} \vec{n}$$

$$\vec{n} \cdot (\vec{v} \cdot \vec{\tau}) = u_t \tau_{tn} + u_n \tau_{nn}$$
$$= U_{st} \tau_{tn} + U_{sn} \tau_{nn}$$

On free surface: if no surface tension $\Longrightarrow \tau_{tn} = 0, -p + \tau_{nn} = -p_0$;
if no viscosity (potential flow) $\Longrightarrow p = p_0$

Energy conservation for liquid motion inside a tank

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\int_{S(t)} (p - \tau_{nn}) U_n dS + \int_{S(t)} U_t \tau_{tn} dS - \int_{Q(t)} \tau_{ij} \frac{\partial u_i}{\partial x_j} dQ$$

where $S(t)$ is wetted surface.

For potential flows under $T = \frac{2\pi}{\sigma}$ oscillations, $\langle \dot{E} \rangle = -\langle \int_{S(t)} p U_n dS \rangle = 0$.
If U_n has $\cos(\sigma t) \Longrightarrow p = a_1 \sin(\sigma t) + \sum_{j=2}^{\infty} a_j \cos(j\sigma t + \varepsilon_j)$

1.

$$\begin{aligned}\rho\vec{v}\cdot\frac{\partial\vec{v}}{\partial t} &= -\rho\vec{v}\cdot(\vec{v}\cdot\nabla)\vec{v} - \vec{v}\cdot\nabla(p+\rho g z) + \vec{v}\cdot\nabla\cdot(\tau_{ij}\vec{e}_i\vec{e}_j)\\ &= -\rho(\vec{v}\cdot\nabla)\left(\frac{\vec{v}\cdot\vec{v}}{2}+\frac{p}{\rho}+gz\right) - \nabla\cdot(\vec{v}\cdot\vec{\tau}) - \tau_{ij}\frac{\partial u_i}{\partial x_j}\\ &= -\nabla\cdot\left[\rho\left(\frac{\vec{v}\cdot\vec{v}}{2}+\frac{p}{\rho}+gz\right)\vec{v} - \vec{v}\cdot\vec{\tau}\right] - \tau_{ij}\frac{\partial u_i}{\partial x_j}\end{aligned}$$

2.

$$\begin{aligned}\vec{v}\cdot(\vec{v}\cdot\nabla)\vec{v} &= \vec{v}\cdot\left[\nabla\frac{\vec{v}\cdot\vec{v}}{2} + \vec{v}\times(\nabla\times\vec{v})\right]\\ &= \vec{v}\cdot\nabla\frac{\vec{v}\cdot\vec{v}}{2} + \vec{v}\cdot[\vec{v}\times(\nabla\times\vec{v})]\\ &= \vec{v}\cdot\nabla\frac{\vec{v}\cdot\vec{v}}{2} + \underbrace{\vec{v}\times\vec{v}}_{=0}\cdot(\nabla\times\vec{v})\\ &= (\vec{v}\cdot\nabla)\frac{\vec{v}\cdot\vec{v}}{2}\end{aligned}$$

3. $\nabla\cdot(f\vec{v}) = f(\underbrace{\nabla\cdot\vec{v}}_{=0}) + \vec{v}\cdot(\nabla f) = (\vec{v}\cdot\nabla)f$

$$(\vec{v}\cdot\nabla)\left(\frac{\vec{v}\cdot\vec{v}}{2}+\frac{p}{\rho}+gz\right) = \nabla\cdot\left[\left(\frac{\vec{v}\cdot\vec{v}}{2}+\frac{p}{\rho}+gz\right)\vec{v}\right]$$

4.

$$\nabla\cdot(\vec{v}\cdot\vec{\tau}) = \begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\cdot\left\{\begin{bmatrix}u & v & w\end{bmatrix}\cdot\begin{bmatrix}\tau_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \tau_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \tau_{zz}\end{bmatrix}\right\}$$

$$\begin{aligned}&= \begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\cdot\begin{bmatrix}u\tau_{xx}+v\tau_{yx}+w\tau_{zx}\\ u\tau_{xy}+v\tau_{yy}+w\tau_{zy}\\ u\tau_{xz}+v\tau_{yz}+w\tau_{zz}\end{bmatrix}^T\\ &= \frac{\partial u\tau_{xx}+v\tau_{yx}+w\tau_{zx}}{\partial x} + \frac{\partial u\tau_{xy}+v\tau_{yy}+w\tau_{zy}}{\partial y} + \frac{\partial u\tau_{xz}+v\tau_{yz}+w\tau_{zz}}{\partial z}\\ &= \frac{\partial u}{\partial x}\tau_{xx} + \frac{\partial v}{\partial x}\tau_{yx} + \frac{\partial w}{\partial x}\tau_{zx} + u\frac{\partial\tau_{xx}}{\partial x} + v\frac{\partial\tau_{yx}}{\partial x} + w\frac{\partial\tau_{zx}}{\partial x} +\\ &\quad \frac{\partial u}{\partial y}\tau_{xy} + \frac{\partial v}{\partial y}\tau_{yy} + \frac{\partial w}{\partial y}\tau_{zy} + u\frac{\partial\tau_{xy}}{\partial y} + v\frac{\partial\tau_{yy}}{\partial y} + w\frac{\partial\tau_{zy}}{\partial y} +\\ &\quad \frac{\partial u}{\partial z}\tau_{xz} + \frac{\partial v}{\partial z}\tau_{yz} + \frac{\partial w}{\partial z}\tau_{zz} + u\frac{\partial\tau_{xz}}{\partial z} + v\frac{\partial\tau_{yz}}{\partial z} + w\frac{\partial\tau_{zz}}{\partial z}\\ &= u_i\frac{\partial\tau_{ij}}{\partial x_j} + \tau_{ij}\frac{\partial u_i}{\partial x_j}\end{aligned}$$

$$
\begin{aligned}
\vec{v}\cdot\nabla\cdot(\tau_{ij}\vec{e}_i\vec{e}_j) &= \begin{bmatrix} u & v & w \end{bmatrix}\cdot \begin{bmatrix} \frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z} \\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z} \\ \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z} \end{bmatrix}^T \\
&= u\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+v\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)+w\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right) \\
&= u_i\frac{\partial\tau_{ij}}{\partial x_j} = \nabla\cdot(\vec{v}\cdot\vec{\tau}) - \tau_{ij}\frac{\partial u_i}{\partial x_j}
\end{aligned}
$$

1.4 NS in RCS

1. $U_g = -\vec{g}\cdot\vec{r} = -(g_1, g_2, g_3)\cdot(x, y, z)$
2. DBC and KBC for Q;
   LDBC and LKBC for Q_0: $\zeta = O(\varepsilon) \Longrightarrow \Phi(x, y, z = \Sigma_0, t) = \Phi(x, y, 0, t)$

1.4.1 RCS

Lagrangian ICS $D*/Dt : x_i = x_i(t) \Longrightarrow$ Eulerian RCS: x_i, t are independent with each other.

1. $\frac{\mathrm{d}\vec{e}_i(t)}{\mathrm{d}t} = \vec{\omega}\times\vec{e}_i(t)$

$$
\begin{cases} \left|\frac{\mathrm{d}\vec{e}_i(t)}{\mathrm{d}t}\right| = \frac{(\omega dt)(\sin\theta)}{dt} = \omega\sin\theta \\ \text{direction: } \frac{\omega\times\vec{e}_i(t)}{|\omega\times\vec{e}_i(t)|} = \frac{\omega\times\vec{e}_i(t)}{\omega\sin\theta} \end{cases} \Longrightarrow
$$

$$
\frac{\mathrm{d}\vec{e}_i(t)}{\mathrm{d}t} = \omega\sin\theta\cdot\frac{\omega\times\vec{e}_i(t)}{\omega\sin\theta} = \vec{\omega}\times\vec{e}_i(t)
$$

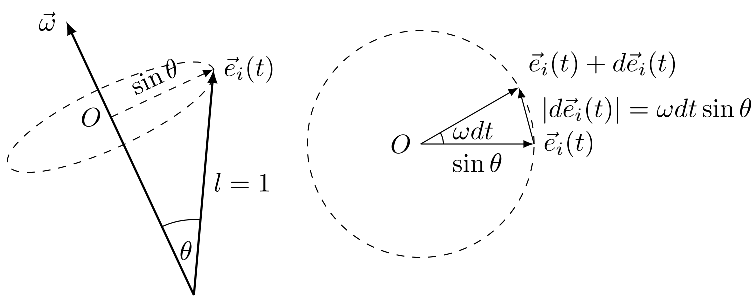

Figure 2: Time derivatives of a vector in a rotating frame of reference.

2. For $\boldsymbol{r}'(t) = \boldsymbol{r}'_O(t) + x_i(t)\vec{e}_i(t)$ in ICS:

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{r}'(t)}{\mathrm{d}t} =& \boldsymbol{r}'_O(t) + \dot{x}(t)\vec{e}_1(t) + \dot{y}(t)\vec{e}_2(t) + \dot{z}(t)\vec{e}_3(t) + x(t)\dot{\vec{e}}_1(t) + y(t)\dot{\vec{e}}_2(t) + z(t)\dot{\vec{e}}_3(t) \\
=& \boldsymbol{r}'_O(t) + \vec{v}_r(t) + \vec{\omega} \times [x(t)\vec{e}_1(t) + y(t)\vec{e}_2(t) + z(t)\vec{e}_3(t)] \\
=& \boldsymbol{r}'_O(t) + v_r(t) + \vec{\omega} \times \vec{r}(t)
\end{aligned}
$$

3. For rigid-body velocity of the tank (fixed (x, y, z) on the tank):

$$
\begin{aligned}
&\boldsymbol{r}'(t) = \boldsymbol{r}'_O(t) + \boldsymbol{r}(t) \\
&\frac{\mathrm{d}\boldsymbol{r}'(t)}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}'_O(t)}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t} \\
&\Longrightarrow \boldsymbol{v}'_b(t) = \boldsymbol{v}'_O(t) + \boldsymbol{\omega} \times \boldsymbol{r}(t)
\end{aligned}
$$

1.4.2 NS in RCS

Lagrangian description follows a fluid particle in space and time: $\boldsymbol{r}'(t) = \boldsymbol{r}'_O(t) + \boldsymbol{r}(t) \Longrightarrow$

$$
\begin{aligned}
\frac{D\boldsymbol{r}'(t)}{Dt} &= \boldsymbol{v}'(t) = \boldsymbol{v}'_O(t) + \vec{v}_r(t) + \vec{\omega} \times \vec{r}(t) \\
\frac{D^2\boldsymbol{r}'(t)}{Dt^2} &= \boldsymbol{a}'(t) \\
&= \boldsymbol{a}'_O(t) + \frac{\mathrm{d}^*\vec{v}_r(t)}{\mathrm{d}t} + \vec{\omega} \times \vec{v}_r(t) + \left[\frac{\mathrm{d}^*\vec{\omega}}{\mathrm{d}t} + \omega \times \omega\right] \times \vec{r}(t) + \vec{\omega} \times [\vec{v}_r(t) + \vec{\omega} \times \vec{r}(t)] \\
&= \boldsymbol{a}'_O(t) + \vec{a}_r(t) + \underbrace{2\vec{\omega} \times \vec{v}_r(t)}_{\text{coriolis acceleration}} + \frac{\mathrm{d}^*\vec{\omega}}{\mathrm{d}t} \times \vec{r}(t) + \underbrace{\vec{\omega} \times [\vec{\omega} \times \vec{r}(t)]}_{\text{centripetal acceleration}}
\end{aligned}
$$

Eulerian description examines the rate of change in time and space at fixed points (x, y, z):

$$
\vec{a}_r(t) = \frac{D^*\vec{v}_r(t)}{Dt} = \frac{\partial^*\vec{v}_r(x, y, z, t)}{\partial t} + \vec{v}_r(x, y, z, t) \cdot \nabla\vec{v}_r(x, y, z, t)
$$

$$
\boldsymbol{r}'(t) = \boldsymbol{r}'_O(t) + \boldsymbol{r}(t)(\text{Lagrangian description}) = \begin{cases} \boldsymbol{r}'(x', y', z')(\text{Eulerian description in ICS}) \\ \boldsymbol{r}'_O(t) + \vec{r}(x, y, z)(\text{Eulerian description in RCS}) \end{cases}
$$

Absolute velocity:

$$
\boldsymbol{v}'(t) = \boldsymbol{v}'_O(t) + \boldsymbol{v}_r(t) + \vec{\omega} \times \vec{r}(t) = \begin{cases} \boldsymbol{v}'(x', y', z', t)(\text{Eulerian description in ICS}) \\ \boldsymbol{v}'_O(t) + \vec{v}_r(x, y, z, t) + \vec{\omega} \times \vec{r}(x, y, z)(\text{Eulerian description in RCS}) \end{cases}
$$

Rigid-body velocity: $\boldsymbol{v}'_b(t) = \boldsymbol{v}'_O(t) + \vec{\omega} \times \vec{r}(x, y, z)$

$\vec{\omega} \times \vec{r}(x, y, z) = \vec{i}[\omega_2(t)z - \omega_3(t)y] - \vec{j}[\omega_1(t)z - \omega_3(t)x] + \vec{k}[\omega_1(t)y - \omega_2(t)x]$
$\Longrightarrow \nabla \cdot (\vec{\omega} \times \vec{r}) = 0$, $\nabla(\vec{\omega} \times \vec{r}) \neq 0$, $\nabla^2(\vec{\omega} \times \vec{r}) = 0$(linear function)

Eulerian NS in ICS $\Longrightarrow$ Eulerian NS in RCS, superpose $O'x'y'z'$ and $Oxyz$ at t:

$$\begin{aligned}\frac{D^2\boldsymbol{r}'(t)}{Dt^2} =&\boldsymbol{a}'_O(t) + \frac{\partial^*\vec{v}_r(x,y,z,t)}{\partial t} + \vec{v}_r(x,y,z,t)\cdot\nabla\vec{v}_r(x,y,z,t) \\ &+ 2\vec{\omega}\times\vec{v}_r(x,y,z,t) + \frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}\times\vec{r}(x,y,z) + \vec{\omega}\times[\vec{\omega}\times\vec{r}(x,y,z)] \\ =& -\frac{1}{\rho}\nabla p + \vec{g} + \nu\nabla^2\vec{v}_r(x,y,z,t)\end{aligned} \tag{8}$$

1.4.3 Governing equations for potential flows in RCS

1. $r'(t) = r'_O(t) + r(t) = r'(x',y',z') = r'_O(t) + r(x,y,z)$ at time t? $\Longrightarrow$ $(x,y,z)-(x',y',z') = \boldsymbol{v}'_b(x,y,z,t)\cdot\Delta t$ at time $t+\Delta t$
2. $\Phi(x',y',z',t) = \Phi(x,y,z,t) \overset{?}{\Longrightarrow} \Phi(x',y',z',t+\Delta t) = \Phi(x,y,z,t+\Delta t)$

Eulerian description in ICS for a fixed point (x',y',z') $\xRightarrow{\text{Transform}}$ Eulerian description in RCS for a fixed point (x,y,z) coinciding with (x',y',z') at time t $\Longrightarrow$

1. **Laplace equation for potential flows in RCS**

$$\nabla\Phi(x',y',z',t) = \nabla^2\Phi(x,y,z,t) = 0 \text{ in } Q(t) \tag{9}$$

2. **Bernoulli's equation for potential flows in RCS**

$$\begin{aligned} &p - p_0 \\ &= -\rho\left[\frac{\partial\Phi(x,y,z,t)}{\partial t} - [\boldsymbol{v}'_O(t) + \vec{\omega}\times\vec{r}(x,y,z)]\cdot\nabla\Phi(x,y,z,t)\right. \\ &\qquad \left. + \frac{[\nabla\Phi(x,y,z,t)]^2}{2} - \vec{g}\cdot\vec{r}(x,y,z)\right]\end{aligned} \tag{10}$$

BE in ICS for fixed point (x',y',z') $\Longrightarrow$

$$p + \rho\left[\frac{\partial\Phi(x',y',z',t)}{\partial t} + \frac{[\nabla\Phi(x',y',z',t)]^2}{2} + U_g\right] = C(t)$$

Let (x,y,z) coincide with (x',y',z') at time t $\Longrightarrow$ $(x,y,z)-(x',y',z') =$

$\boldsymbol{v}'_b(x,y,z,t)\Delta t = [\boldsymbol{v}'_O(t) + \boldsymbol{\omega} \times \boldsymbol{r}(x,y,z,t)]\Delta t$ at time $t+\Delta t \Longrightarrow$

$$\begin{aligned}
&\frac{\partial \Phi(x,y,z,t)}{\partial t} \\
&= \lim_{\Delta t \to 0} \frac{\Phi(x,y,z,t+\Delta t) - \Phi(x,y,z,t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\overbrace{[\Phi(x',y',z',t+\Delta t) + \nabla\Phi(x',y',z',t+\Delta t)\cdot \boldsymbol{v}'_b(x,y,z,t)\Delta t]}^{\text{Taylor expansion}} - \Phi(x',y',z',t)}{\Delta t} \\
&= \frac{\partial \Phi(x',y',z',t)}{\partial t} + \boldsymbol{v}'_b(x,y,z,t)\nabla\Phi(x',y',z',t) \\
&= \frac{\partial \Phi(x',y',z',t)}{\partial t} + \boldsymbol{v}'_b(x,y,z,t)\nabla\Phi(x,y,z,t) \\
&\Longrightarrow \boxed{\frac{\partial \Phi(x',y',z',t)}{\partial t} = \frac{\partial \Phi(x,y,z,t)}{\partial t} - \boldsymbol{v}'_b(x,y,z,t)\nabla\Phi(x,y,z,t)}
\end{aligned}$$

$Oxyz$ coincides with $Ox'y'z'$ at $t \Longrightarrow U_g = -\boldsymbol{g}\cdot\boldsymbol{r}'(x',y',z') = -\boldsymbol{g}\cdot\boldsymbol{r}(x,y,z)$

3. Boundary conditions on wetted surface $S(t)$: $\vec{v}_r \cdot \vec{n} = 0, \boldsymbol{v}' \cdot \vec{n} = \boldsymbol{v}'_b \cdot \vec{n} \Longrightarrow$

$$\frac{\partial \Phi(x,y,z,t)}{\partial n} = \boldsymbol{v}' \cdot \vec{n} = \boldsymbol{v}'_O(t)\cdot\vec{n} + [\vec{\omega} \times \vec{r}(x,y,z)]\cdot\vec{n} \tag{11}$$

4. Dynamic free-surface condition $p = p_0$ on $\Sigma(t)$:

$$\frac{\partial \Phi(x,y,z,t)}{\partial t} - [\boldsymbol{v}'_O(t) + \vec{\omega}\times\vec{r}(x,y,z)]\cdot\nabla\Phi(x,y,z,t) + \frac{[\nabla\Phi(x,y,z,t)]^2}{2} - \vec{g}\cdot\vec{r}(x,y,z) = 0 \tag{12}$$

5. Kinetic free-surface condition:

$$\begin{aligned}
&-\frac{\partial \Phi(x,y,z,t)}{\partial x}\frac{\partial \zeta(x,y,t)}{\partial x} - \frac{\partial \Phi(x,y,z,t)}{\partial x}\frac{\partial \zeta(x,y,t)}{\partial x} + \frac{\partial \Phi(x,y,z,t)}{\partial z} \\
&= [\boldsymbol{v}'_O(t) + \boldsymbol{\omega} \times \boldsymbol{r}(x,y,z)] \cdot \left(-\frac{\partial \zeta(x,y,t)}{\partial x}, -\frac{\partial \zeta(x,y,t)}{\partial y}, 1\right) + \frac{\partial \zeta(x,y,t)}{\partial t}
\end{aligned} \tag{13}$$

The fluid particles remain on the free surface for the entire time: $Z(t) = 0 \Longrightarrow$

$$\begin{aligned}
0 = \frac{DZ(t)}{Dt} &= \frac{\partial Z(x',y',z',t)}{\partial t} + \boldsymbol{v}'(x',y',z',t)\cdot\nabla Z(x',y',z',t) \\
&= \left[\frac{\partial Z(x,y,z,t)}{\partial t} - \boldsymbol{v}'_b(x,y,z,t)\cdot\nabla Z(x,y,z,t)\right] + \nabla\Phi(x,y,z,t)\cdot\nabla Z(x,y,z,t)
\end{aligned}$$

$\boxed{Z(x,y,z,t) = z - \zeta(x,y,t) = 0} \Longrightarrow \nabla Z(x,y,z,t) = \left(-\frac{\partial\zeta(x,y,t)}{\partial x}, -\frac{\partial\zeta(x,y,t)}{\partial y}, 1\right), \frac{\partial Z(x,y,z,t)}{\partial t} =$

$-\frac{\partial\zeta(x,y,t)}{\partial t} \Longrightarrow$

$$\left(\frac{\partial\Phi(x,y,z,t)}{\partial x},\frac{\partial\Phi(x,y,z,t)}{\partial y},\frac{\partial\Phi(x,y,z,t)}{\partial z}\right)\cdot\left(-\frac{\partial\zeta(x,y,t)}{\partial x},-\frac{\partial\zeta(x,y,t)}{\partial y},1\right)$$
$$=[\boldsymbol{v}'_O(t)+\boldsymbol{\omega}\times\boldsymbol{r}(x,y,z)]\cdot\left(-\frac{\partial\zeta(x,y,t)}{\partial x},-\frac{\partial\zeta(x,y,t)}{\partial y},1\right)+\frac{\partial\zeta(x,y,t)}{\partial t}$$

$\vec{n}=\frac{\nabla Z(x,y,z,t)}{|\nabla Z(x,y,z,t)|}$ on $\Sigma(t) \Longrightarrow$

$$\frac{\partial\Phi(x,y,z,t)}{\partial n}=\boldsymbol{v}'_b(x,y,z,t)\cdot\vec{n}-\frac{\frac{\partial Z(x,y,z,t)}{\partial t}}{|\nabla Z(x,y,z,t)|}$$
$$=[\boldsymbol{v}'_O(t)+\vec{\omega}\times\vec{r}(x,y,z)]\cdot\vec{n}+\frac{\frac{\partial\zeta(x,y,t)}{\partial t}}{\sqrt{\left(\frac{\partial\zeta(x,y,t)}{\partial x}\right)^2+\left(\frac{\partial\zeta(x,y,t)}{\partial y}\right)^2+1}}$$

6. LKBC and LDBC

For free oscillations: $\boldsymbol{v}'_O(t)=\boldsymbol{\omega}=0 \Longrightarrow$

$$\text{LKBC: }\frac{\partial\Phi(x,y,0,t)}{\partial z}=\frac{\partial\zeta(x,y,t)}{\partial t} \tag{14}$$

$$\text{LDBC: }\zeta(x,y,t)=-\frac{1}{g}\frac{\partial\Phi(x,y,0,t)}{\partial t} \tag{15}$$

$$\text{Combined LBC: }\frac{\partial^2\Phi(x,y,0,t)}{\partial t^2}+g\frac{\partial\Phi(x,y,0,t)}{\partial z}=0 \tag{16}$$

For forced oscillations:

$$\text{LKBC: }\frac{\partial\Phi(x,y,0,t)}{\partial n}=\frac{\partial\Phi(x,y,0,t)}{\partial z}=[\boldsymbol{v}'_O(t)+\vec{\omega}\times\vec{r}(x,y,0)]\cdot\vec{n}+\frac{\partial\zeta(x,y,t)}{\partial t} \tag{17}$$

$$\text{LDBC: }\frac{\partial\Phi(x,y,0,t)}{\partial t}-g_1x-g_2y-g_3\zeta(x,y,t)=0 \tag{18}$$

$\frac{\eta_{1,2,3}}{L},\eta_{4,5,6}=O(\varepsilon)$

$\zeta(x,y,t)$(for free oscillations), $\frac{\zeta(x,y,t)}{L}$, $\frac{\partial\zeta(x,y,t)}{\partial x}$, $\frac{\partial\zeta(x,y,t)}{\partial y}$(surface slope) $=O(\varepsilon)$

$o(\varepsilon)$ terms are neglected in DBC and KBC $\Longrightarrow$

$\Phi(x,y,z=\Sigma(t))=\Phi(x,y,z=\Sigma_0=0)$

$\vec{n}=\frac{\nabla Z(x,y,z,t)}{|\nabla Z(x,y,z,t)|}=\frac{\left(\frac{\partial\zeta}{\partial x},\frac{\partial\zeta}{\partial y},1\right)}{\sqrt{\cancel{\left(\frac{\partial\zeta(x,y,t)}{\partial x}\right)^2}+\cancel{\left(\frac{\partial\zeta(x,y,t)}{\partial y}\right)^2}+1}}$

For free oscillations:

LKBC:

$$-\cancel{\frac{\partial\Phi(x,y,z,t)}{\partial x}\frac{\partial\zeta(x,y,t)}{\partial x}}-\cancel{\frac{\partial\Phi(x,y,z,t)}{\partial x}\frac{\partial\zeta(x,y,t)}{\partial x}}+\frac{\partial\Phi(x,y,0,t)}{\partial z}$$

$$=\underbrace{\cancel{[\boldsymbol{v}'_O(t)+\boldsymbol{\omega}\times\boldsymbol{r}(x,y,z)]}}_{=0 \text{ for free osscillations}}\cdot\left(-\frac{\partial\zeta(x,y,t)}{\partial x},-\frac{\partial\zeta(x,y,t)}{\partial y},1\right)+\frac{\partial\zeta(x,y,t)}{\partial t}$$

LDBC:

$$\frac{\partial\Phi(x,y,0,t)}{\partial t}-\underbrace{\cancel{[\boldsymbol{v}'_O(t)+\boldsymbol{\omega}\times\boldsymbol{r}(x,y,z)]}}_{=0 \text{ for free oscillations}}\cdot\nabla\Phi(x,y,z,t)+\cancel{\frac{[\nabla\Phi(x,y,z,t)]^2}{2}}-(0,0,-g)\cdot(x,y,\zeta)=0$$

For forced oscillations:

LKBC:

$$\frac{\partial\Phi(x,y,0,t)}{\partial n}=\left(\frac{\partial\Phi}{\partial x},\frac{\partial\Phi}{\partial y},\frac{\partial\Phi}{\partial z}\right)\cdot\left(-\frac{\partial\zeta}{\partial x},-\frac{\partial\zeta}{\partial y},1\right)=\frac{\partial\Phi(x,y,0,t)}{\partial z}$$

$$=[\boldsymbol{v}'_O(t)+\vec{\omega}\times\vec{r}(x,y,0)]\cdot\vec{n}+\frac{\frac{\partial\zeta(x,y,t)}{\partial t}}{\sqrt{\cancel{\left(\frac{\partial\zeta(x,y,t)}{\partial x}\right)^2}+\cancel{\left(\frac{\partial\zeta(x,y,t)}{\partial y}\right)^2}+1}}$$

LDBC:

$$\frac{\partial\Phi(x,y,0,t)}{\partial t}-\underbrace{\cancel{[\boldsymbol{v}'_O(t)+\vec{\omega}\times\vec{r}(x,y,z)]\cdot\nabla\Phi(x,y,z,t)}}_{=O(\varepsilon^2) \text{ for forced oscillations}}+\cancel{\frac{[\nabla\Phi(x,y,z,t)]^2}{2}}-(g_1,g_2,g_3)\cdot(x,y,\zeta)=0$$

7. Mass(volume) conservation condition

$$\int_{\Sigma(t)}\zeta(x,y,t)dxdy=0 \tag{19}$$

2 Linear Natural Sloshing Modes

Some formulas:

1. $e^{ikx}=\cos(kx)+i\sin(kx)$
2. $\left(e^{ikx}\right)'=ike^{ikx}$
3. $\nabla\cdot(\Phi\nabla\psi)=\Phi\nabla^2\psi+\nabla\Phi\cdot\nabla\psi$

2.1 Natural frequencies and modes for a 2D Rectangular Tank

1. Time-periodic solutions of linearized free oscillations with circular frequency ω:

$$\begin{cases} \Phi(x,y,z,t) = \dfrac{ig}{\omega}\varphi(x,y,z)e^{i\omega t} = \dfrac{ig}{\omega}\varphi(x,y,z)[\cos(\omega t) + i\sin(\omega t)], \quad i^2 = -1 \\ \zeta(x,y,t) = -\dfrac{1}{g}\dfrac{\partial \Phi(x,y,0,t)}{\partial t}(\text{LDBC}) = \varphi(x,y,0)e^{i\omega t} = f(x,y)e^{i\omega t} = f(x,y)[\cos(\omega t) + i\sin(\omega t)] \end{cases} \tag{20}$$

2. Substitute $\Phi(x,y,z,t)$ and $\zeta(x,y,t)$ into linearized boundary-value problem in RCS $\Longrightarrow$ spectral boundary problem:

$$\begin{cases} \nabla^2\Phi(x,y,z,t) = 0 \\ \dfrac{\partial \Phi(x,y,z,t)}{\partial n} = \nabla\Phi(x,y,z,t)\cdot\vec{n} = 0 \\ \dfrac{\partial^2 \Phi(x,y,z,t)}{\partial t^2} + g\dfrac{\partial \Phi(x,y,z,t)}{\partial z} = 0 \\ \displaystyle\int_{\Sigma_0} \zeta(x,y,t)dxdy = 0 \end{cases} \Longrightarrow \begin{cases} \nabla^2\varphi(x,y,z) = 0 \text{ in } Q_0 \\ \dfrac{\partial \varphi(x,y,z)}{\partial n} = \nabla\varphi(x,y,z)\cdot\vec{n} = 0 \text{ on } S_0 \\ \dfrac{\partial \varphi(x,y,z)}{\partial z} - \underbrace{\left(\kappa = \dfrac{\omega^2}{g}\right)}_{\text{eigenvalues}} \underbrace{\varphi(x,y,z)}_{\text{eigenfunctions}} = 0 \text{ on } \Sigma_0 \\ \displaystyle\int_{\Sigma_0} \varphi(x,y,0)dxdy = 0 \end{cases} \tag{21}$$

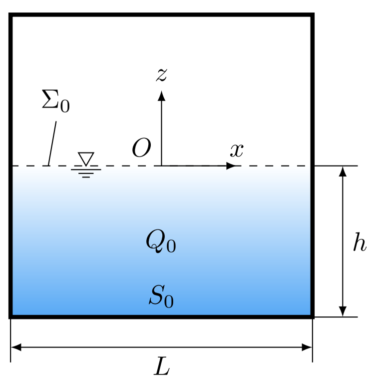

Figure 3: 2D rectangular tank.

3. Spectral boundary problem for a 2D rectangular tank:

$$\begin{cases} \dfrac{\partial^2 \varphi(x,z)}{\partial x^2} + \dfrac{\partial^2 \varphi(x,z)}{\partial z^2} = 0 \text{ in } Q_0 \\ \dfrac{\partial \varphi(x,z)}{\partial x} = 0 \text{ on } x = \pm \dfrac{L}{2} \text{ for } -h \le z \le 0 \\ \dfrac{\partial \varphi(x,z)}{\partial z} = 0 \text{ on } z = -h \text{ for } -\dfrac{L}{2} \le x \le \dfrac{L}{2} \\ \dfrac{\partial \varphi(x,z)}{\partial z} - \kappa \varphi(x,z) = 0 \text{ on } z = 0 \text{ for } -\dfrac{L}{2} \le x \le \dfrac{L}{2} \\ \displaystyle\int_{-\frac{L}{2}}^{\frac{L}{2}} \varphi(x,0) dx = 0 \end{cases} \tag{22}$$

4.

Linear natural sloshing modes

Eigenvalues: $\kappa_n = \frac{\pi n}{L} \tanh\Big(\frac{\pi n}{L} h\Big)$ (23)

Eigenfunctions (Linear natural sloshing modes):

$$\varphi_n(x,z) = \cos\left[\frac{\pi n}{L}\left(x + \frac{L}{2}\right)\right] \cdot \frac{\cosh\left[\frac{\pi n}{L}(z+h)\right]}{\cosh\left(\frac{\pi n}{L} h\right)} \tag{24}$$

Wave patterns: $f_n(x) = \varphi(x,0) = \cos\left[\frac{\pi n}{L}\left(x + \frac{L}{2}\right)\right]$ (25)

$$\begin{aligned}
&\int_{-\frac{L}{2}}^{\frac{L}{2}} f_n(x) f_m(x) dx \\
&= \int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left[\frac{\pi n}{L}\left(x + \frac{L}{2}\right)\right] \cos\left[\frac{\pi m}{L}\left(x + \frac{L}{2}\right)\right] dx \\
&= \frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left[\frac{\pi}{L}\left(x + \frac{L}{2}\right)(n+m)\right] + \cos\left[\frac{\pi}{L}\left(x + \frac{L}{2}\right)(n-m)\right] dx \\
&= \underbrace{\frac{1}{2}\left(\frac{\sin\left[\frac{\pi}{L}\left(x+\frac{L}{2}\right)(n+m)\right]}{\frac{\pi n}{L}(n+m)}\right)_{-\frac{L}{2}}^{\frac{L}{2}}}_{=0} \\
&\quad + \frac{1}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left[\frac{\pi}{L}\left(x + \frac{L}{2}\right)(n-m)\right] dx \\
&= \begin{cases} 0 \text{ if } n \neq m \\ \frac{L}{2} \text{ if } n = m \end{cases}
\end{aligned}$$

Natural frequencies: $\omega_n = \sqrt{g\kappa_n} = \sqrt{g\frac{\pi n}{L}\tanh\Big(\frac{\pi n}{L} h\Big)}$ (26)

Natural periods: $T_n = \frac{2\pi}{\sqrt{g\frac{\pi n}{L}\tanh\left(\frac{\pi n}{L} h\right)}}, \quad n = 1, 2, \ldots$ (27)

Solutions of spectral problem (22): $\varphi(x,z) = X(x)Z(z) \Longrightarrow$

(a) $\frac{\partial^2\varphi(x,z)}{\partial x^2}+\frac{\partial^2\varphi(x,z)}{\partial z^2}=0$ in $Q_0 \Longrightarrow$

$$\frac{\frac{\mathrm{d}^2X(x)}{\mathrm{d}x^2}}{X(x)}=-\frac{\frac{\mathrm{d}^2Z(z)}{\mathrm{d}z^2}}{Z(z)}=C_1<0$$

$$\Longrightarrow \begin{cases} X(x)=C_2\cos\sqrt{-C_1}x+C_3\sin\sqrt{-C_1}x \\ Z(z)=C_4e^{\sqrt{-C_1}z}+C_5e^{-\sqrt{-C_1}z} \end{cases}$$

From (4d,4b,4c): $\sqrt{-C_1}=\dfrac{n\pi}{L}, C_4=C_5e^{2\sqrt{-C_1}h}$

$$\Longrightarrow \begin{cases} \kappa_n=\dfrac{n\pi}{L}\tanh\Big(\dfrac{n\pi}{h}\Big) \\ X(x)=C_6\cos\left[\dfrac{n\pi}{L}\left(x+\dfrac{L}{2}\right)\right] \\ Z(z)=C_5e^{2\frac{n\pi}{L}h+\frac{n\pi}{L}z}+C_5e^{-\frac{n\pi}{L}z} \\ \qquad =2e^{\frac{n\pi}{L}h}C_5\dfrac{e^{2\left(\frac{n\pi}{L}h+\frac{n\pi}{L}z\right)}+1}{2e^{\frac{n\pi}{L}h+\frac{n\pi}{L}z}} \\ \qquad =2e^{\frac{n\pi}{L}h}C_5\cosh\left[\dfrac{n\pi}{L}(z+h)\right] \\ \qquad =C_7\dfrac{\cosh\left[\frac{n\pi}{L}(z+h)\right]}{\cosh\left(\frac{n\pi}{L}h\right)} \end{cases}$$

(b) $\frac{\partial \varphi(x,z)}{\partial x} = 0$ on $x = \pm\frac{L}{2}$ for $-h \leq z \leq 0 \Longrightarrow$

$$
\begin{aligned}
&Z(z)\frac{\mathrm{d}X(x)}{\mathrm{d}x} = 0 \\
&\Longrightarrow -C_2\cancel{\sqrt{-C_1}}\sin\left(\sqrt{-C_1}x\right) + C_3\cancel{\sqrt{-C_1}}\cos\left(\sqrt{-C_1}x\right)\Big|_{x=\pm\frac{L}{2}} = 0 \\
&\Longrightarrow \begin{cases} -C_2\sin\left(\dfrac{L\sqrt{-C_1}}{2}\right) + C_3\cos\left(\dfrac{L\sqrt{-C_1}}{2}\right) = 0 \\ C_2\sin\left(\dfrac{L\sqrt{-C_1}}{2}\right) + C_3\cos\left(\dfrac{L\sqrt{-C_1}}{2}\right) = 0 \end{cases} \\
&\Longrightarrow \begin{cases} 2C_2\sin\left(\dfrac{L\sqrt{-C_1}}{2}\right) = 0 \\ 2C_3\cos\left(\dfrac{L\sqrt{-C_1}}{2}\right) = 0 \end{cases} \\
&\Longrightarrow \text{From (4e), } C_2 = 0 \text{ or } \sin\left(\dfrac{L\sqrt{-C_1}}{2}\right) = 0 \\
&\Longrightarrow \begin{cases} C_2 = 0, \cos\left(\dfrac{L\sqrt{-C_1}}{2}\right) = 0 \Longrightarrow \dfrac{l\sqrt{-C_1}}{2} = \dfrac{(2n-1)\pi}{2} \Longrightarrow \sqrt{-C_1} = \dfrac{(2n-1)\pi}{L} \\ C_3 = 0, \sin\left(\dfrac{L\sqrt{-C_1}}{2}\right) = 0 \Longrightarrow \dfrac{L\sqrt{-C_1}}{2} = n\pi \Longrightarrow \sqrt{-C_1} = \dfrac{2n\pi}{L} \end{cases} \\
&\Longrightarrow \begin{cases} X(x) = C_2\cos\left(\dfrac{2n\pi}{L}x\right) = (-1)^n C_2\cos\left[\dfrac{2n\pi}{L}\left(x + \dfrac{L}{2}\right)\right] \\ X(x) = C_3\sin\left(\dfrac{(2n-1)\pi}{L}x\right) = (-1)^n C_3\cos\left(\dfrac{(2n-1)\pi}{L}(x + \dfrac{L}{2})\right) \end{cases} \\
&\Longrightarrow \begin{cases} X(x) = C_6\cos\left(\dfrac{n\pi}{L}(x + \dfrac{L}{2})\right) \\ \sqrt{-C_1} = \dfrac{n\pi}{L} \end{cases}
\end{aligned}
$$

(c) $\frac{\partial \varphi}{\partial z} = 0$ on $z = -h$ for $|x| \leq \frac{L}{2} \Longrightarrow$

$$
\begin{aligned}
&X(x)\frac{\mathrm{d}Z(z)}{\mathrm{d}z} = 0 \\
&\Longrightarrow C_4\cancel{\sqrt{-C_1}}e^{\sqrt{-C_1}z} - C_5\cancel{\sqrt{-C_1}}\, e^{-\sqrt{-C_1}z}\Big|_{z=-h} = 0 \\
&\Longrightarrow C_4e^{-\sqrt{-C_1}h} - C_5e^{\sqrt{-C_1}h} = 0, C_4 \neq 0, C_5 \neq 0 \\
&\Longrightarrow \frac{C_4}{C_5} = e^{2\sqrt{-C_1}h}
\end{aligned}
$$

(d) $\frac{\partial\varphi(x,z)}{\partial z} - \kappa\varphi(x,z) = 0$ on $z = 0$ for $-\frac{L}{2} \leq x \leq \frac{L}{2} \implies$

$$-\kappa X(x)Z(z) + X(x)\frac{\mathrm{d}Z(z)}{\mathrm{d}z} = 0$$
$$\implies -\kappa\left(C_4 e^{\sqrt{-C_1}z} + C_5 e^{-\sqrt{-C_1}z}\right)\Big|_{z=0} + \left(C_4\sqrt{-C_1}e^{\sqrt{-C_1}z} - C_5\sqrt{-C_1}e^{-\sqrt{-C_1}z}\right)\Big|_{z=0} = 0$$
$$\implies \left(\sqrt{-C_1} - \kappa\right)C_4 - \left(\sqrt{-C_1} + \kappa\right)C_5 = 0$$

From (4b,4c), $\sqrt{-C_1} = \dfrac{n\pi}{L}$ and $\dfrac{C_4}{C_5} = e^{2\sqrt{-C_1}h}$

$$\implies \kappa = \sqrt{-C_1}\frac{\frac{C_4}{C_5} - 1}{\frac{C_4}{C_5} + 1} = \frac{n\pi}{L}\frac{e^{2\frac{n\pi h}{L}} - 1}{e^{2\frac{n\pi h}{L}} + 1} = \frac{n\pi}{L}\tanh\left(\frac{n\pi}{h}\right)$$

(e) $\int_{-\frac{L}{2}}^{\frac{L}{2}} \varphi(x,0)dx = 0 \implies$

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} X(x)Z(0)dx = 0$$
$$\implies (C_4 + C_5)\, 2\frac{C_2}{\sqrt{-C_1}}\sin\frac{L\sqrt{-C_1}}{2} = 0$$
$$\implies C_2 = 0 \text{ or } \sin\left(\frac{L\sqrt{-C_1}}{2}\right) = 0$$

5. $\tanh\left(\frac{\pi n}{L}h\right) \approx \frac{\pi n}{L}h$.

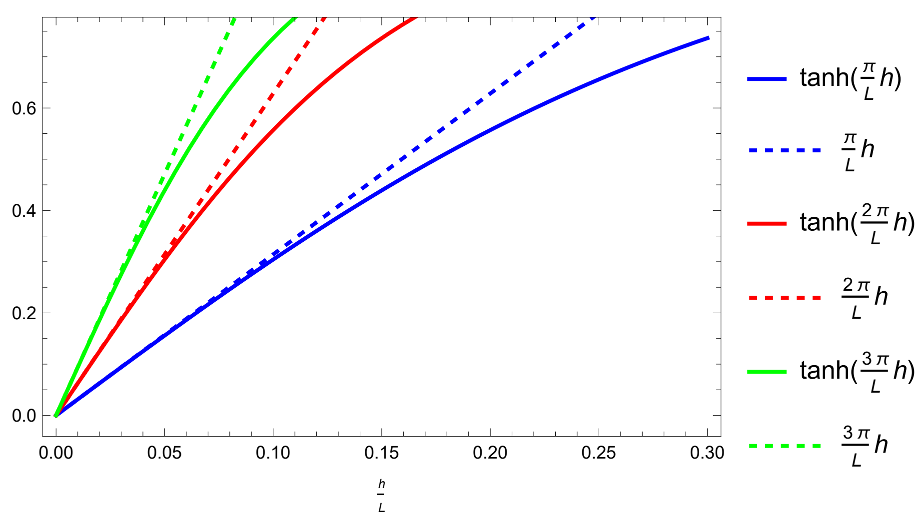

Figure 4: Approximation of tanh function.

3 Linear Modal Theory

3.1 Linear modal equations

1. Linearized boundary-value problem

$$
\begin{cases}
\dfrac{\partial^2\Phi(x,z)}{\partial x^2} + \dfrac{\partial^2\Phi(x,z)}{\partial z^2} = 0 \text{ in } Q_0 \\
\dfrac{\partial\Phi(x,z)}{\partial n} = [\boldsymbol{v}'_O(t) + \vec{\omega}\times\vec{r}(x,z)]\cdot\vec{n} \\
\qquad = \boldsymbol{v}'_O(t)\cdot\vec{n} + [\vec{\omega}\times\vec{r}(x,z)]\cdot\vec{n} = \boldsymbol{v}'_O(t)\cdot\vec{n} + \vec{\omega}\cdot[\vec{r}(x,z)\times\vec{n}] \\
\qquad = \boldsymbol{v}'_O(t)\cdot\vec{n} + \dot{\eta}_5(t)\cdot(zn_1 - xn_3) \text{ on } S_0 \\
\dfrac{\partial\Phi(x,0)}{\partial n} = \dfrac{\partial\Phi(x,0)}{\partial z} \\
\qquad = [\boldsymbol{v}'_O(t) + \vec{\omega}\times\vec{r}(x,0)]\cdot\vec{n} + \dfrac{\partial\zeta(x,t)}{\partial t} \\
\qquad = \boldsymbol{v}'_O(t)\cdot\vec{n} + [\vec{\omega}\times\vec{r}(x,0)]\cdot\vec{n} + \dfrac{\partial\zeta(x,t)}{\partial t} = \boldsymbol{v}'_O(t)\cdot\vec{n} + \vec{\omega}\cdot[\vec{r}(x,0)\times\vec{n}] + \dfrac{\partial\zeta(x,t)}{\partial t} \\
\qquad = \boldsymbol{v}'_O(t)\cdot\vec{n} + \dot{\eta}_5(t)\cdot(0n_1 - xn_3) + \dfrac{\partial\zeta(x,t)}{\partial t} \text{ on } \Sigma_0 \\
\dfrac{\partial\Phi(x,0)}{\partial t} - g_1 x - g_3\zeta(x,t) = \dfrac{\partial\Phi(x,0)}{\partial t} - g\eta_5(t)x + g\zeta(x,t) = 0 \text{ on } \Sigma_0 \\
\displaystyle\int_{\Sigma_0}\zeta(x,t)dx = 0
\end{cases}
\tag{28}
$$

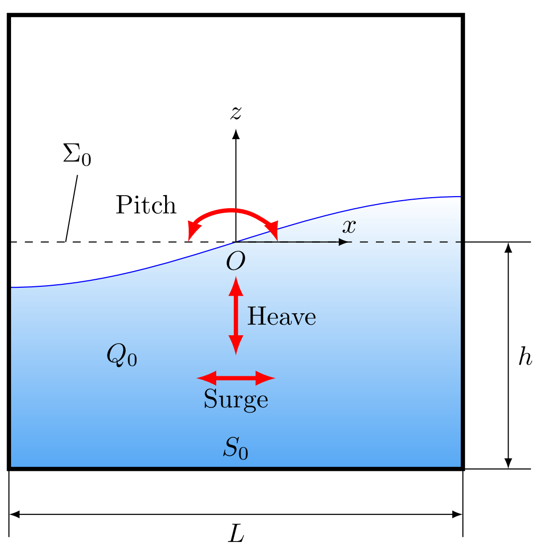

Figure 5: Linearized boundary-value problem in a 2D rectangular tank

2. **Represent $\zeta(x,t)$ and $\Phi(x,z,t)$ with the infinite set of generalized coordinates $\beta_i(t)$ and $R_i(t)$**

$$\begin{cases} \zeta(x,t) = \sum_{i=1}^{\infty} \beta_i(t)\varphi_i(x,0) = \sum_{i=1}^{\infty} \beta_i(t) f_i(x) \\ \Phi(x,z) = \boldsymbol{v}'_O(t) \cdot \vec{r}(x,z) + \vec{\omega}(t) \cdot \vec{\Omega}_0(x,z) + \sum_{i=1}^{\infty} R_i(t)\varphi_i(x,z) \end{cases} \tag{29}$$

Stokes-Joukowski potential:

$\vec{\omega} \cdot \frac{\partial \vec{\Omega}_0}{\partial n} = \vec{\omega} \cdot \nabla \vec{\Omega}_0 \cdot \vec{n} = (\vec{\omega} \times \vec{r}) \cdot \vec{n} = \vec{\omega} \cdot (\vec{r} \times \vec{n})$ on $S_0 \cup \Sigma_0 \Longrightarrow$

$\vec{\omega} \cdot \nabla \vec{\Omega}_0 = \vec{\omega} \times \vec{r}$

$$\frac{\partial \vec{\Omega}_0}{\partial n} = \nabla \vec{\Omega}_0 \cdot \vec{n} = \vec{r} \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ n_1 & n_2 & n_3 \end{vmatrix}$$
$$= \vec{i}(yn_3 - zn_2) + \vec{j}(zn_1 - xn_3) + \vec{k}(xn_2 - yn_1)$$

$\vec{\Omega}_0(x,z) = (0, \vec{\Omega}_{02}(x,z), 0)$ for 2D liquid motions in $x - z$ plane

$$\begin{cases} \nabla^2 \Omega_{02}(x,z) = 0 \text{ in } Q_0 \\ \dfrac{\partial \Omega_{02}(x,z)}{\partial n} = \nabla \Omega_{02}(x,z) \cdot \vec{n} = (z\vec{i} - x\vec{k}) \cdot \vec{n} = zn_1 - xn_3 \\ \qquad \text{on } S_0 \cup \Sigma_0 \end{cases} \tag{30}$$

For linearized boundary-value problem (28): Laplace equation, wetted surface condition and volume conservation condition are automatically satisfied $\Longrightarrow$ LKBC and LDBC $\Longrightarrow \beta_i(t), R_i(t)$.

3. Substitute $\zeta(x,t), \Phi(x,z,t)$ into LKBC $\Longrightarrow$

$$\frac{\partial \Phi(x,0,t)}{\partial n} = \nabla \Phi(x,0,t) \cdot \vec{n} = \frac{\partial \Phi(x,0,t)}{\partial z}$$

$$= \boldsymbol{v}'_O(t) \cdot \nabla \vec{r}(x,0) \cdot \vec{n} + \vec{\omega}(t) \cdot \frac{\partial \vec{\Omega}_0(x,0)}{\partial n} + \sum_{i=1}^{\infty} R_i(t) \frac{\partial \varphi_i(x,0)}{\partial z} \text{ on } \Sigma_0$$

$$= \boldsymbol{v}'_O(t) \cdot \vec{n} + \vec{\omega}(t) \cdot [\vec{r}(x,0) \times \vec{n}] + \sum_{i=1}^{\infty} R_i(t) \kappa_i \varphi_i(x,0)$$

$$= \boldsymbol{v}'_O(t) \cdot \vec{n} + \vec{\omega}(t) \cdot [\vec{r}(x,0) \times \vec{n}] + \sum_{i=1}^{\infty} \frac{\mathrm{d}\beta_i(t)}{\mathrm{d}t} f_i(x)$$

$$\Longrightarrow \sum_{i=1}^{\infty} R_i(t) \kappa_i \varphi_i(x,0) = \sum_{i=1}^{\infty} R_i(t) \kappa_i f(x) = \sum_{i=1}^{\infty} \frac{\mathrm{d}\beta_i(t)}{\mathrm{d}t} f_i(x)$$

$$\Longrightarrow \sum_{i=1}^{\infty} \int_{\Sigma_0} R_i(t) \kappa_i f_i(x) f_j(x) dS = \sum_{i=1}^{\infty} \int_{\Sigma_0} \frac{\mathrm{d}\beta_i(t)}{\mathrm{d}t} f_i(x) f_j(x) dS$$

$$\Longrightarrow \boxed{\frac{\mathrm{d}\beta_j(t)}{\mathrm{d}t} = \kappa_j R_j(t) \text{ for } j \geq 1}$$

4. Substitute $\zeta(x,t), \Phi(x,z,t)$, and $\frac{\mathrm{d}\beta_j(t)}{\mathrm{d}t} = \kappa_j R_j(t)$ into LDBC
   $U_g = -(g_1, g_3) \cdot (x, \zeta) = -g_1 x - g_3 \zeta$
   $\vec{r}|_{\Sigma_0} = (x, 0)$ or $\vec{r}|_{\Sigma_0} = (x, \zeta)$?
   $\vec{r}|_{\Sigma_0} = (x, 0) \Longrightarrow$ heave oscillation can not linearly excite sloshing?

Linear modal equations (infinite set of uncoupled linear differential equations for the generalized coordinates $\{\beta_i(t)\}$)

$$\frac{\mathrm{d}\beta_j(t)}{\mathrm{d}t} = \kappa_j R_j(t), \quad j = 1, 2, \ldots \tag{31}$$

$$\begin{aligned}\frac{\mathrm{d}^2\beta_j(t)}{\mathrm{d}t^2} &+ \omega_j^2 \left(1 + \frac{\ddot{\eta}_3(t)}{g}\right) \beta_j(t) \\ &= -\frac{\lambda_{1j}}{\mu_j} \cdot [\ddot{\eta}_1(t) - g\eta_5(t)] - \frac{\lambda_{02j}}{\mu_j} \cdot \ddot{\eta}_5(t), \quad j = 1, 2, \ldots\end{aligned} \tag{32}$$

where $\lambda_{1j} = \int_{-\frac{L}{2}}^{\frac{L}{2}} x f_j(x) dx$

$$\lambda_{02j} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \Omega_{02}(x, 0) f_j(x) dx$$

$$\mu_j = \frac{\int_{-\frac{L}{2}}^{\frac{L}{2}} f_j^2(x) dx}{\kappa_j}$$

$$\frac{\partial\Phi(x,0)}{\partial t} - g\eta_5(t)x + g\zeta(x,t) = 0 \text{ on } \Sigma_0$$

$$\Longrightarrow \frac{\mathrm{d}\boldsymbol{v}'_O(t)}{\mathrm{d}t}\cdot\vec{r}(x,0 \text{ or } \zeta) + \ddot{\eta}_5(t)\cdot\Omega_{02}(x,0) + \sum_{i=1}^{\infty}\frac{\mathrm{d}R_i(t)}{\mathrm{d}t}\varphi_i(x,0) - g\eta_5(t)x + g\sum_{i=1}^{\infty}\beta_i(t)f_i(x) = 0$$

$$\Longrightarrow \ddot{\eta}_1(t)x + \ddot{\eta}_3(t)\cdot(0 \text{ or } \zeta) + \ddot{\eta}_5(t)\cdot\Omega_{02}(x,0) + \sum_{i=1}^{\infty}\frac{1}{\kappa_i}\frac{\mathrm{d}^2\beta_i(t)}{\mathrm{d}t^2}f_i(x) - g\eta_5(t)x + \frac{\omega_i^2}{\kappa_i}\sum_{i=1}^{\infty}\beta_i(t)f_i(x) = 0$$

$$\xRightarrow{\vec{r}=(x,0)}$$

$$\sum_{i=1}^{\infty}\frac{1}{\kappa_i}\left(\frac{\mathrm{d}^2\beta_i(t)}{\mathrm{d}t^2} + \omega_i^2\beta_i(t)\right)f_i(x) + x(\ddot{\eta}_1(t) - g\eta_5(t)) + 0\ddot{\eta}_3(t) + \ddot{\eta}_5(t)\Omega_{02}(x,0) = 0$$

$$\Longrightarrow \sum_{i=1}^{\infty}\left(\frac{\mathrm{d}^2\beta_i(t)}{\mathrm{d}t^2} + \omega_i^2\beta_i(t)\right)\frac{\int_{-\frac{L}{2}}^{\frac{L}{2}} f_i(x)f_j(x)dx}{\kappa_i} + (\ddot{\eta}_1(t) - g\eta_5(t))\int_{-\frac{L}{2}}^{\frac{L}{2}} xf_j(x)dx$$

$$+ \ddot{\eta}_5(t)\int_{-\frac{L}{2}}^{\frac{L}{2}}\Omega_{02}(x,0)f_j(x)dx = 0$$

$$\Longrightarrow \left(\frac{\mathrm{d}^2\beta_j(t)}{\mathrm{d}t^2} + \omega_j^2\beta_j(t)\right)\frac{\int_{-\frac{L}{2}}^{\frac{L}{2}} f_j^2(x)dx}{\kappa_j} + (\ddot{\eta}_1(t) - g\eta_5(t))\int_{-\frac{L}{2}}^{\frac{L}{2}} xf_j(x)dx$$

$$+ \ddot{\eta}_5(t)\int_{-\frac{L}{2}}^{\frac{L}{2}}\Omega_{02}(x,0)f_j(x)dx = 0$$

$$\xRightarrow{\vec{r}=(x,\zeta)}$$

$$\sum_{i=1}^{\infty}\frac{1}{\kappa_i}\left[\frac{\mathrm{d}^2\beta_i(t)}{\mathrm{d}t^2} + \omega_i^2\beta_i(t)\right]f_i(x) + \ddot{\eta}_3(t)\frac{\omega_i^2/g}{\kappa_i}\sum_{i=1}^{\infty}\beta_i(t)f_i(x) + x(\ddot{\eta}_1(t) - g\eta_5(t)) + \ddot{\eta}_5(t)\Omega_{02}(x,0) = 0$$

$$\sum_{i=1}^{\infty}\frac{1}{\kappa_i}\left[\frac{\mathrm{d}^2\beta_i(t)}{\mathrm{d}t^2} + \omega_i^2\left(1 + \frac{\ddot{\eta}_3(t)}{g}\right)\beta_i(t)\right]f_i(x) + x(\ddot{\eta}_1(t) - g\eta_5(t)) + \ddot{\eta}_5(t)\Omega_{02}(x,0) = 0$$

$$\Longrightarrow \left[\frac{\mathrm{d}^2\beta_j(t)}{\mathrm{d}t^2} + \omega_j^2\left(1 + \frac{\ddot{\eta}_3(t)}{g}\right)\beta_j(t)\right]\frac{\int_{-\frac{L}{2}}^{\frac{L}{2}} f_j^2(x)dx}{\kappa_j} + (\ddot{\eta}_1(t) - g\eta_5(t))\int_{-\frac{L}{2}}^{\frac{L}{2}} xf_j(x)dx$$

$$+ \ddot{\eta}_5(t)\int_{-\frac{L}{2}}^{\frac{L}{2}}\Omega_{02}(x,0)f_j(x)dx = 0$$

5. Linear pressure $p = p_0 + \rho\vec{g}\cdot\vec{r}(x,z) - \rho\frac{\partial\Phi(x,z,t)}{\partial t} \Longrightarrow$

Linear pressure

$$p = p_0 + \rho g \left(x\eta_5(t) - z \right) - \rho \left[\ddot{\eta}_1(t)x + \ddot{\eta}_3(t)z + \ddot{\eta}_5(t) \cdot \Omega_{02}(x,z) + \sum_{i=1}^{\infty} \frac{1}{\kappa_i} \frac{\mathrm{d}^2\beta_i(t)}{\mathrm{d}t^2} \varphi_i(x,z) \right] \quad (33)$$

3.2 Forced pitch(η_5) and heave(η_3) sloshing in a 2D rectangular tank

3.2.1 Linear modal equations for coupled pitch-heave sloshing

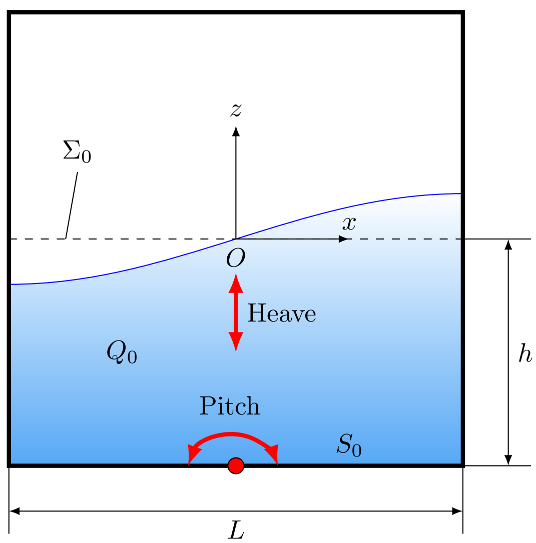

Figure 6: Forced pitch(η_5) and heave(η_3) sloshing in a 2D rectangular tank.

Linear modal equations for coupled pitch-heave sloshing in a 2D rectangular tank

$$\frac{\mathrm{d}\beta_n(t)}{\mathrm{d}t} = \kappa_n R_n(t), \quad n = 1, 2, \ldots$$

$$\begin{aligned}
&\frac{\mathrm{d}^2\beta_n(t)}{\mathrm{d}t^2} + \omega_n^2 \left(1 + \frac{\ddot{\eta}_3(t)}{g}\right) \beta_n(t) \\
&= -\frac{\lambda_{1n}}{\mu_n} \cdot \left[\ddot{\eta}_1(t) - g\eta_5(t) + \frac{\lambda_{02n}}{\lambda_{1n}} \cdot \ddot{\eta}_5(t)\right] \\
&= -\frac{\lambda_{1n}}{\mu_n} \cdot \left[\left(h + \frac{\lambda_{02n}}{\lambda_{1n}}\right) \cdot \ddot{\eta}_5(t) - g\eta_5(t)\right] \\
&= -P_n \cdot \left[(h + S_n) \cdot \ddot{\eta}_5(t) - g\eta_5(t)\right], \quad n = 1, 2, \ldots
\end{aligned}$$

where $P_n = \dfrac{2\left[(-1)^n - 1\right] \tanh\left(\frac{\pi h n}{L}\right)}{\pi n}$

$S_n = -\dfrac{2L \tanh\left(\frac{\pi h n}{2L}\right)}{\pi n}$

1.

$$\lambda_{1n} = \int_{-\frac{L}{2}}^{\frac{L}{2}} x f_n(x) dx = \int_{-\frac{L}{2}}^{\frac{L}{2}} x \cos\left[\frac{\pi n}{L}\left(x + \frac{L}{2}\right)\right] dx = \frac{L^2}{\pi^2} \frac{(-1)^n - 1}{n^2}$$

$$\lambda_{02n} = -\frac{4L^2}{\pi^3} \frac{(-1)^n - 1}{n^3} \tanh\left(\frac{\pi n}{L} \frac{h}{2}\right) \int_{-\frac{L}{2}}^{\frac{L}{2}} f_n^2(x) dx = -\frac{2L^3}{\pi^3} \frac{(-1)^n - 1}{n^3} \tanh\left(\frac{\pi n}{L} \frac{h}{2}\right)$$

$$\mu_n = \frac{\int_{-\frac{L}{2}}^{\frac{L}{2}} f_n^2(x) dx}{\kappa_n} = \frac{L/2}{\frac{\pi n}{L} \tanh\left(\frac{\pi n}{L} h\right)} = \frac{L^2}{2\pi} \frac{\coth\left(\frac{\pi n}{L} h\right)}{n}$$

2. $\eta_1(t) = h\eta_5(t)$

3. Stokes-Joukowski potential $\Omega_{02}(x, z)$:

$$\begin{cases}
\dfrac{\partial^2 \Omega_{02}(x, z)}{\partial x^2} + \dfrac{\partial^2 \Omega_{02}(x, z)}{\partial z^2} = 0 \text{ in } Q_0 \\
\dfrac{\partial \Omega_{02}(x, z)}{\partial x} = z \text{ on } x = \pm\dfrac{L}{2} \text{ for } -h \le z \le 0 \\
\dfrac{\partial \Omega_{02}(x, z)}{\partial z} = -x \text{ on } z = -h, 0 \text{ for } -\dfrac{L}{2} \le x \le \dfrac{L}{2}
\end{cases}$$

$\Omega_{02}(x,z) = zx + F(x,z) \Longrightarrow$

$$\begin{cases} \dfrac{\partial^2 F(x,z)}{\partial x^2} + \dfrac{\partial^2 F(x,z)}{\partial z^2} = 0 \\ \dfrac{\partial F(x,z)}{\partial x} = 0 \text{ on } x = \pm\dfrac{L}{2} \text{ for } -h \leq z \leq 0 \\ \dfrac{\partial F(x,z)}{\partial z} = -2x \text{ on } z = -h, 0 \text{ for } -\dfrac{L}{2} \leq x \leq \dfrac{L}{2} \end{cases}$$

$F(x,z) = X(x)Z(z) \Longrightarrow$

(a) $\frac{\partial^2 F(x,z)}{\partial x^2} + \frac{\partial^2 F(x,z)}{\partial z^2} = 0$ in $Q_0 \Longrightarrow$

$$\frac{\frac{\mathrm{d}X(x)}{\mathrm{d}x}}{X(x)} = -\frac{\frac{\mathrm{d}Z(z)}{\mathrm{d}z}}{Z(z)} = C_1 < 0$$

$$\Longrightarrow \begin{cases} X(x) = C_2 \cos\left(\sqrt{-C_1}x\right) + C_3 \sin\left(\sqrt{-C_1}x\right) \\ Z(z) = C_4 e^{\sqrt{-C_1}z} + C_5 e^{-\sqrt{-C_1}z} \end{cases}$$

(b) $\frac{\partial F(x,z)}{\partial x} = 0$ on $x = \pm\frac{L}{2}$ for $-h \le z \le 0 \Longrightarrow$

$$\begin{cases} -C_2 \sin\dfrac{\sqrt{-C_1}L}{2} + C_3\cos\dfrac{\sqrt{-C_1}L}{2} = 0 \\ C_2 \sin\dfrac{\sqrt{-C_1}L}{2} + C_3\cos\dfrac{\sqrt{-C_1}L}{2} = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} C_2 \sin\dfrac{\sqrt{-C_1}L}{2} = 0 \\ C_3\cos\dfrac{\sqrt{-C_1}L}{2} = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} C_2 = 0, \cos\dfrac{\sqrt{-C_1}L}{2} = 0 \Longrightarrow \sqrt{-C_1} = \dfrac{(2k+1)\pi}{L} \\ \qquad \Longrightarrow X(x) = C_3 \sin\left[\dfrac{(2k+1)\pi}{L}x\right] = -C_3\cos\left[\dfrac{(2k+1)\pi}{L}\left(x+\dfrac{L}{2}\right)\right] \\ C_3 = 0, \sin\dfrac{\sqrt{-C_1}L}{2} = 0 \Longrightarrow \sqrt{-C_1} = \dfrac{2\kappa\pi}{L} \\ \qquad \Longrightarrow X(x) = C_2\cos\left(\dfrac{2k\pi}{L}x\right) = (-1)^k C_2 \cos\left[\dfrac{2k\pi}{L}\left(x+\dfrac{L}{2}\right)\right] \end{cases}$$

$$\Longrightarrow \begin{cases} \sqrt{-C_1} = \dfrac{n\pi}{L} \\ X_n(x) = C_6\cos\left[\dfrac{n\pi}{L}\left(x+\dfrac{L}{2}\right)\right] = C_6 f_n(x) \\ Z_n(z) = C_4 e^{\frac{n\pi}{L}z} + C_5 e^{-\frac{n\pi}{L}z} \end{cases}$$

$$\Longrightarrow F_n(x,z) = X_n(x)Z_n(z) = f_n(y)\cdot\left(C_{1n}e^{\frac{n\pi}{L}z} + C_{2n}e^{-\frac{n\pi}{L}z}\right)$$

$$\Longrightarrow F(x,z) = \sum_{n=1}^{\infty} f_n(x)\cdot\left(C_{1n}e^{\frac{n\pi z}{L}} + C_{2n}e^{-\frac{n\pi z}{L}}\right)$$

$$\Longrightarrow F(x,z) = \sum_{n=1}^{\infty} C_n f_n(x)\frac{\sinh\left[\frac{\pi n}{L}\left(z+\frac{h}{2}\right)\right]}{\cosh\left(\frac{\pi n}{L}\frac{h}{2}\right)}$$

(c) $\frac{\partial F(x,z)}{\partial z} = -2x$ on $z = 0$ for $-\frac{L}{2} \le x \le \frac{L}{2} \Longrightarrow$

$$\sum_{n=1}^{\infty} C_n \frac{\pi n}{L} f_n(x) = -2x$$

$$\Longrightarrow \sum_{n=1}^{\infty} C_n \frac{\pi n}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f_n(x) f_m(x)dx = -2\int_{-\frac{L}{2}}^{\frac{L}{2}} x f_m(x)dx$$

$$\Longrightarrow C_m \frac{\pi m}{L}\frac{L}{2} = -2\frac{L^2\left[(-1)^m - 1\right]}{\pi^2 m^2}$$

$$\Longrightarrow C_m = -\frac{4L^2}{\pi^3}\frac{(-1)^m - 1}{m^3}$$

(d) **Solutions of Stokes-Joukowski potential for sloshing in a 2D rectangular tank**

$$\Omega_{02}(x,z) = xz - \frac{4L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^3}f_n(x)\frac{\sinh\left[\frac{\pi n}{L}\left(z+\frac{h}{2}\right)\right]}{\cosh\left(\frac{\pi n}{L}\frac{h}{2}\right)} \quad (34)$$

3.2.2 Solutions of linear modal equations for pitch sloshing

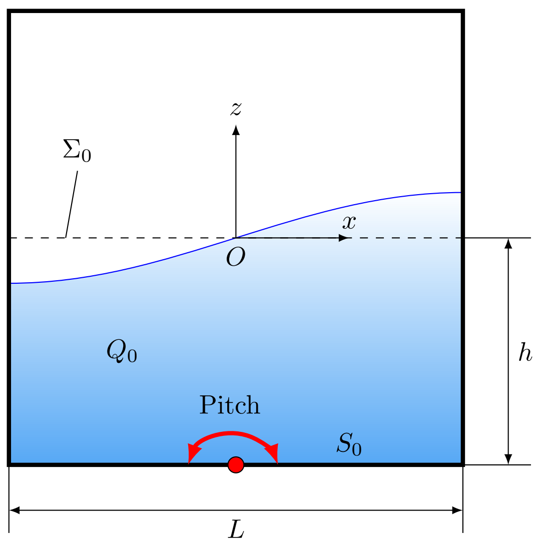

Figure 7: Forced pitch(η_5) sloshing in a 2D rectangular tank.

$\eta_5(t) = A_p \sin(\omega_p t) \Longrightarrow$

$$\frac{\mathrm{d}^2\beta_n(t)}{\mathrm{d}t^2} + \omega_n^2\beta_n(t) = -P_n\cdot\left[(h+S_n)\cdot\ddot{\eta}_5(t) - g\eta_5(t)\right] = P_n\cdot\left[(h+S_n)\,\omega_p^2+g\right]A_p\sin(\omega_p t)$$

4 Multimodal method

1. Proof of the natural conditions of the Bateman-Luke formulation.
2. Proof of the relationship between the original boundary-value problem and Bateman-Luke system.
3. Linear modal equations $\Longleftrightarrow$ Nonlinear modal equations

4.1 Variational principle

Bateman-Luke principle

$$W(Z,\Phi) = \int_{t_1}^{t_2} L dt$$

$$L = \int_{Q(t)} (p - p_0) dQ \xRightarrow{\text{Bernoulli's eqn in RCS}}$$

$$- \rho \int_{Q(t)} \left[\frac{\partial \Phi(x,y,z,t)}{\partial t} - [\boldsymbol{v}'_O(t) + \vec{\omega} \times \vec{r}(x,y,z)] \cdot \nabla\Phi(x,y,z,t) \right.$$

$$\left. + \frac{[\nabla\Phi(x,y,z,t)]^2}{2} - \vec{g} \cdot \vec{r}(x,y,z) \right] dQ$$

$$\delta\Phi(x,y,z,t_1) = 0, \quad \delta\Phi(x,y,z,t_2) = 0$$

$$\delta Z(x,y,z,t_1) = 0, \quad \delta Z(x,y,z,t_2) = 0 \tag{35}$$

4.2 Modal system based on the Bateman-Luke formulation

Modal representations of the free surface and velocity potential:

$$\begin{cases} \zeta(x,t) = \sum_{i=1}^{\infty} \beta_i(t)\varphi_i(x,0) = \sum_{i=1}^{\infty} \beta_i(t) f_i(x) \\ \Phi(x,z,t) = \boldsymbol{v}'_O(t) \cdot \vec{r}(x,z) + \vec{\omega}(t) \cdot \vec{\Omega}(x,z,t) + \sum_{i=1}^{\infty} R_i(t)\varphi_i(x,z) \end{cases}$$

$$\begin{cases} \nabla^2\Omega_2(x,z,t) = 0 \text{ in } Q(t) \\ \dfrac{\partial\Omega_2(x,z,t)}{\partial n} = \nabla\Omega_2(x,z,t) \cdot \vec{n} = zn_1 - xn_3 \text{ on } S(t) \cup \Sigma(t) \end{cases}$$

Nonlinear modal equations

$$
\begin{cases}
\sum_{i=1}^{\infty} \frac{\partial D_n}{\partial \beta_i} \dot{\beta}_i - \sum_{k=1}^{\infty} D_{nk} R_k = 0, \quad n = 1, 2, \ldots \text{ (Kinematics)} \\
- \sum_{n=1}^{\infty} \left(\dot{R}_n \frac{\partial D_n}{\partial \beta_i} \right) - \frac{1}{2} \sum_{n,k=1}^{\infty} \left(R_n R_k \frac{\partial D_{nk}}{\partial \beta_i} \right) \\
- \ddot{\eta}_5(t) \frac{\partial l_{2\omega}}{\partial \beta_i} - \dot{\eta}_5(t) \frac{\partial l_{2\omega t}}{\partial \beta_i} + \frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{\eta}_5(t) \frac{\partial l_{2\omega t}}{\partial \dot{\beta}_i} \right) \\
\left[-\ddot{\eta}_1(t) - \dot{\eta}_5(t) \ddot{\eta}_3(t) + g_1 \right] \frac{\partial l_1}{\partial \beta_i} + \left[-\ddot{\eta}_3(t) + \dot{\eta}_5(t) \ddot{\eta}_1(t) + g_3 \right] \frac{\partial l_3}{\partial \beta_i} \\
+ \frac{1}{2} \left[\dot{\eta}_5(t) \right]^2 \frac{\partial J_{22}^1}{\partial \beta_i} = 0, \quad i = 1, 2, \ldots \text{ (Dynamic boundary condition)}
\end{cases}
\tag{36}
$$

Nonlinear modal equations accounting for the asymptotic relation

$$
\begin{cases}
O(A_{\text{Horizontal motion}}) = O(A_{\text{Rotational motion}}) = \epsilon \\
O(\beta_1) = O(\epsilon^{\frac{1}{3}}) \\
O(\beta_2) = O(\epsilon^{\frac{2}{3}}) \\
O(\beta_3) = O(\epsilon) \\
\text{Higher order terms than } O(\epsilon) \text{ are neglected in the nonlinear euqation.}
\end{cases}
\tag{37}
$$

$$
\begin{cases}
\sum_{i=1}^{\infty} \frac{\partial D_n}{\partial \beta_i} \dot{\beta}_i - \sum_{k=1}^{\infty} D_{nk} R_k = 0, \quad n = 1, 2, \ldots \text{ (Kinematics)} \\
- \sum_{n=1}^{\infty} \left(\dot{R}_n \frac{\partial D_n}{\partial \beta_i} \right) - \frac{1}{2} \sum_{n,k=1}^{\infty} \left(R_n R_k \frac{\partial D_{nk}}{\partial \beta_i} \right) \\
- \ddot{\eta}_5(t) \frac{\partial l_{2\omega}}{\partial \beta_i} - \dot{\eta}_5(t) \frac{\partial l_{2\omega t}}{\partial \beta_i} + \frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{\eta}_5(t) \frac{\partial l_{2\omega t}}{\partial \dot{\beta}_i} \right) \\
\left[-\ddot{\eta}_1(t) + g_1 \right] \lambda_{1i} + \left[-\ddot{\eta}_3(t) + g_3 \right] \beta_i \lambda_{3i} = 0, \quad i = 1, 2, \ldots \text{ (Dynamics BC)}
\end{cases}
\tag{38}
$$

The pressure-integral Lagrangian L

$$
\begin{aligned}
L(R_n, \dot{R}_n, \beta_i, \dot{\beta}_i) = & -\ddot{\eta}_1(t) \cdot l_1 - \ddot{\eta}_3(t) \cdot l_3 - \dot{\eta}_5(t) \cdot l_{2\omega t} - \ddot{\eta}_5(t) \cdot l_{2\omega} \\
& - \dot{\eta}_5(t)\ddot{\eta}_3(t) \cdot l_1 + \dot{\eta}_5(t)\ddot{\eta}_1(t) \cdot l_3 + \frac{(\dot{\eta}_1(t))^2 + (\dot{\eta}_3(t))^2}{2} \cdot M_l \\
& + \frac{1}{2} [\dot{\eta}_5(t)]^2 J^1_{22} + L_r \\
L_r = & -\sum_{n=1}^{\infty} \frac{\mathrm{d}R_n(t)}{\mathrm{d}t} \cdot D_n - \frac{1}{2} \sum_{n,k=1}^{\infty} R_n(t)R_k(t) \cdot D_{nk} + g_1 \cdot l_1 + g_3 \cdot l_3
\end{aligned}
\tag{39}
$$

where
$$
\begin{aligned}
l_1(\beta_i) &= \rho \int_{Q(t)} x dQ \\
l_3(\beta_i) &= \rho \int_{Q(t)} z dQ \\
\frac{\partial l_1}{\partial \beta_i} &= \rho \int_{\Sigma_0} x f_i(x) dS = \rho \int_{-\frac{L}{2}}^{\frac{L}{2}} x f_i(x) dx = \lambda_{1i} \\
\frac{\partial l_3}{\partial \beta_i} &= \rho \int_{\Sigma_0} [f_i(x)]^2 dS \beta_i = \rho \int_{-\frac{L}{2}}^{\frac{L}{2}} [f_i(x)]^2 dx \beta_i = \lambda_{3i} \beta_i \\
l_{2\omega t}(\beta_i, \dot{\beta}_i) &= \rho \int_{Q(t)} \frac{\partial \Omega_2(x,z,t)}{\partial t} dQ \\
l_{2\omega}(\beta_i) &= \rho \int_{Q(t)} \Omega_2(x,z,t) dQ \\
D_n(\beta_i) &= \rho \int_{Q(t)} \varphi_n(x,z) dQ \\
D_{nk}(\beta_i) &= \rho \int_{Q(t)} [\nabla \varphi_n(x,z) \cdot \nabla \varphi_k(x,z)] \, dQ \\
M_l &= \rho \int_{Q(t)} dQ = \text{constant}
\end{aligned}
$$

1.

$$\delta W=\int_{t_1}^{t_2}\delta L dt=\int_{t_1}^{t_2}-\sum_{n=1}^{\infty}\left(D_n\delta\dot{R}_n\right)-\sum_{n,k=1}^{\infty}\left(D_{nk}R_k\delta R_n\right)$$

$$+\left\{\left[-\ddot{\eta}_1(t)-\dot{\eta}_5(t)\ddot{\eta}_3(t)+g_1\right]\frac{\partial l_1}{\partial\beta_i}+\left[-\ddot{\eta}_3(t)+\dot{\eta}_5(t)\ddot{\eta}_1(t)+g_3\right]\frac{\partial l_3}{\partial\beta_i}-\dot{\eta}_5(t)\frac{\partial l_{2\omega t}}{\partial\beta_i}-\ddot{\eta}_5(t)\frac{\partial l_{2\omega}}{\partial\beta_i}\right.$$

$$\left.+\frac{1}{2}\left[\dot{\eta}_5(t)\right]^2\frac{\partial J_{22}^1}{\partial\beta_i}-\sum_{n=1}^{\infty}\left(\dot{R}_n\frac{\partial D_n}{\partial\beta_i}\right)-\frac{1}{2}\sum_{n,k=1}^{\infty}\left(R_nR_k\frac{\partial D_{nk}}{\partial\beta_i}\right)\right\}\delta\beta_i-\left(\dot{\eta}_5(t)\frac{\partial l_{2\omega t}}{\partial\dot{\beta}_i}\right)\delta\dot{\beta}_i dt=0$$

$$=\int_{t_1}^{t_2}\left[\sum_{n=1}^{\infty}\frac{\partial D_n}{\partial\beta_i}\dot{\beta}_i-\sum_{n,k=1}^{\infty}D_{nk}R_k\right]\delta R_n$$

$$+\left\{\left[-\ddot{\eta}_1(t)-\dot{\eta}_5(t)\ddot{\eta}_3(t)+g_1\right]\frac{\partial l_1}{\partial\beta_i}+\left[-\ddot{\eta}_3(t)+\dot{\eta}_5(t)\ddot{\eta}_1(t)+g_3\right]\frac{\partial l_3}{\partial\beta_i}-\dot{\eta}_5(t)\frac{\partial l_{2\omega t}}{\partial\beta_i}-\ddot{\eta}_5(t)\frac{\partial l_{2\omega}}{\partial\beta_i}\right.$$

$$\left.+\frac{1}{2}\left[\dot{\eta}_5(t)\right]^2\frac{\partial J_{22}^1}{\partial\beta_i}-\sum_{n=1}^{\infty}\left(\dot{R}_n\frac{\partial D_n}{\partial\beta_i}\right)-\frac{1}{2}\sum_{n,k=1}^{\infty}\left(R_nR_k\frac{\partial D_{nk}}{\partial\beta_i}\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\eta}_5(t)\frac{\partial l_{2\omega t}}{\partial\dot{\beta}_i}\right)\right\}\delta\beta_i dt=0$$

2. $\delta R_n(t_1)=\delta R_n(t_2)=\delta\beta_i(t_1)=\delta\beta_i(t_2)=0\Longrightarrow$

$$\int_{t_1}^{t_2}-\sum_{n=1}^{\infty}\left(D_n\delta\dot{R}_n\right)dt=-\sum_{n=1}^{\infty}\int_{t_1}^{t_2}D_n d\delta R_n=-\sum_{n=1}^{\infty}\left[\left(D_n\delta R_n\right)_{t_1}^{t_2}-\int_{t_1}^{t_2}\delta R_n\frac{\partial D_n}{\partial\beta_i}\frac{\mathrm{d}\beta_i}{\mathrm{d}t}dt\right]$$

$$=\int_{t_1}^{t_2}\left(\sum_{n=1}^{\infty}\delta R_n\frac{\partial D_n}{\partial\beta_i}\dot{\beta}_i\right)dt$$

$$\int_{t_1}^{t_2}-\left(\dot{\eta}_5(t)\frac{\partial l_{2\omega t}}{\partial\dot{\beta}_i}\right)\delta\dot{\beta}_i dt=\int_{t_1}^{t_2}-\left(\dot{\eta}_5(t)\frac{\partial l_{2\omega t}}{\partial\dot{\beta}_i}\right)d\delta\beta_i=\left[-\left(\dot{\eta}_5(t)\frac{\partial l_{2\omega t}}{\partial\dot{\beta}_i}\right)\delta\beta_i\right]_{t_1}^{t_2}-\int_{t_1}^{t_2}-\delta\beta_i d\left(\dot{\eta}_5(t)\frac{\partial l_{2\omega t}}{\partial\dot{\beta}_i}\right)$$

$$=\int_{t_1}^{t_2}\delta\beta_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\eta}_5(t)\frac{\partial l_{2\omega t}}{\partial\dot{\beta}_i}\right)dt$$

3.

$$\delta L = L(R_n + \delta R_n, \dot{R}_n + \delta \dot{R}_n, \beta_i + \delta\beta_i, \dot{\beta}_i + \delta\dot{\beta}_i) - L(R_n, \dot{R}_n, \beta_i, \dot{\beta}_i)$$

$$= -\ddot{\eta}_1(t)\cdot\left(\frac{\partial l_1}{\partial\beta_i}\delta\beta_i\right) - \ddot{\eta}_3(t)\cdot\left(\frac{\partial l_3}{\partial\beta_i}\delta\beta_i\right) - \dot{\eta}_5(t)\cdot\left(\frac{\partial l_{2\omega t}}{\partial\beta_i}\delta\beta_i + \frac{\partial l_{2\omega t}}{\partial\dot{\beta}_i}\delta\dot{\beta}_i\right) - \ddot{\eta}_5(t)\cdot\left(\frac{\partial l_{2\omega}}{\partial\beta_i}\delta\beta_i\right)$$

$$- \dot{\eta}_5(t)\ddot{\eta}_3(t)\cdot\left(\frac{\partial l_1}{\partial\beta_i}\delta\beta_i\right) + \dot{\eta}_5(t)\ddot{\eta}_1(t)\cdot\left(\frac{\partial l_3}{\partial\beta_i}\delta\beta_i\right) + \cancel{\frac{(\dot{\eta}_1(t))^2 + (\dot{\eta}_3(t))^2}{2}}\cdot M_l + \frac{1}{2}\left[\dot{\eta}_5(t)\right]^2\cdot\left(\frac{\partial J_{22}^1}{\partial\beta_i}\delta\beta_i\right)$$

$$- \sum_{n=1}^{\infty}\left[\delta\dot{R}_n\cdot D_n + \dot{R}_n\cdot\left(\frac{\partial D_n}{\partial\beta_i}\delta\beta_i\right)\right] - \frac{1}{2}\sum_{n,k=1}^{\infty}\left[2R_k\delta R_n\cdot D_{nk} + R_nR_k\cdot\left(\frac{\partial D_{nk}}{\partial\beta_i}\delta\beta_i\right)\right]$$

$$+ g_1\cdot\left(\frac{\partial l_1}{\partial\beta_i}\delta\beta_i\right) + g_3\cdot\left(\frac{\partial l_3}{\partial\beta_i}\delta\beta_i\right)$$

$$= -\sum_{n=1}^{\infty}\left(D_n\delta\dot{R}_n\right) - \sum_{n,k=1}^{\infty}(D_{nk}R_k\delta R_n)$$

$$+ \Bigg\{\left[-\ddot{\eta}_1(t) - \dot{\eta}_5(t)\ddot{\eta}_3(t) + g_1\right]\frac{\partial l_1}{\partial\beta_i} + \left[-\ddot{\eta}_3(t) + \dot{\eta}_5(t)\ddot{\eta}_1(t) + g_3\right]\frac{\partial l_3}{\partial\beta_i} - \dot{\eta}_5(t)\cdot\left(\frac{\partial l_{2\omega t}}{\partial\beta_i}\right) - \ddot{\eta}_5(t)\frac{\partial l_{2\omega}}{\partial\beta_i}$$

$$+ \frac{1}{2}\left[\dot{\eta}_5(t)\right]^2\frac{\partial J_{22}^1}{\partial\beta_i} - \sum_{n=1}^{\infty}\left(\dot{R}_n\frac{\partial D_n}{\partial\beta_i}\right) - \frac{1}{2}\sum_{n,k=1}^{\infty}\left(R_nR_k\frac{\partial D_{nk}}{\partial\beta_i}\right)\Bigg\}\delta\beta_i - \left(\dot{\eta}_5(t)\frac{\partial l_{2\omega t}}{\partial\dot{\beta}_i}\right)\delta\dot{\beta}_i$$

4.

$$L=-\rho\int_{Q(t)}\left[\frac{\partial\Phi(x,z,t)}{\partial t}-[\boldsymbol{v}_O'(t)+\vec{\omega}\times\vec{r}(x,z)]\cdot\nabla\Phi(x,z,t)+\frac{[\nabla\Phi(x,z,t)]^2}{2}-\vec{g}\cdot\vec{r}(x,z)\right]dQ$$

$$=-\rho\int_{Q(t)}\frac{\mathrm{d}\boldsymbol{v}_O'(t)}{\mathrm{d}t}\cdot\vec{r}(x,z)+\frac{\partial}{\partial t}\left(\vec{\omega}(t)\cdot\vec{\Omega}(x,z,t)\right)+\frac{\partial\varphi(x,z,t)}{\partial t}$$

$$-[\boldsymbol{v}_O'(t)+\vec{\omega}(t)\times\vec{r}(x,z)]\cdot\left[\boldsymbol{v}_O'(t)+\nabla\left(\vec{\omega}(t)\cdot\vec{\Omega}(x,z,t)\right)+\nabla\varphi(x,z,t)\right]$$

$$+\frac{\left[\boldsymbol{v}_O'(t)+\nabla\left(\vec{\omega}(t)\cdot\vec{\Omega}(x,z,t)\right)+\nabla\varphi(x,z,t)\right]^2}{2}-\vec{g}\cdot\vec{r}(x,z)dQ$$

$$=-\rho\int_{Q(t)}\frac{\mathrm{d}\boldsymbol{v}_O'(t)}{\mathrm{d}t}\cdot\vec{r}(x,z)+\frac{\partial}{\partial t}\left[\vec{\omega}(t)\cdot\vec{\Omega}(x,z,t)\right]+\frac{\partial\varphi(x,z,t)}{\partial t}$$

$$-[\boldsymbol{v}_O'(t)]^2\cancel{-\boldsymbol{v}_O'(t)\cdot\nabla\left[\vec{\omega}(t)\cdot\vec{\Omega}(x,z,t)\right]}\cancel{-\boldsymbol{v}_O'(t)\cdot\nabla\varphi(x,z,t)}$$

$$-\vec{\omega}(t)\times\vec{r}(x,z)\cdot\boldsymbol{v}_O'(t)-\vec{\omega}(t)\times\vec{r}(x,z)\cdot\nabla\left[\vec{\omega}(t)\cdot\vec{\Omega}(x,z,t)\right]-\vec{\omega}(t)\times\vec{r}(x,z)\cdot\nabla\varphi(x,z,t)$$

$$+\frac{[\boldsymbol{v}_O'(t)]^2}{2}+\frac{\left[\nabla\left(\vec{\omega}(t)\cdot\vec{\Omega}(x,z,t)\right)\right]^2}{2}+\frac{[\nabla\varphi(x,z,t)]^2}{2}$$

$$\cancel{+\boldsymbol{v}_O'(t)\cdot\nabla\left[\vec{\omega}(t)\cdot\vec{\Omega}(x,z,t)\right]}\cancel{+\boldsymbol{v}_O'(t)\cdot\nabla\varphi(x,z,t)}+\nabla\left[\vec{\omega}(t)\cdot\vec{\Omega}(x,z,t)\right]\cdot\nabla\varphi(x,z,t)$$

$$-\vec{g}\cdot\vec{r}(x,z)dQ$$

$$=-\rho\int_{Q(t)}\frac{\mathrm{d}\boldsymbol{v}_O'(t)}{\mathrm{d}t}\cdot\vec{r}(x,z)+\frac{\partial}{\partial t}\left[\vec{\omega}(t)\cdot\vec{\Omega}(x,z,t)\right]$$

$$-\vec{\omega}(t)\times\vec{r}(x,z)\cdot\boldsymbol{v}_O'(t)-\vec{\omega}(t)\times\vec{r}(x,z)\cdot\nabla\left[\vec{\omega}(t)\cdot\vec{\Omega}(x,z,t)\right]\cancel{-\vec{\omega}(t)\times\vec{r}(x,z)\cdot\nabla\varphi(x,z,t)}$$

$$-\frac{[\boldsymbol{v}_O'(t)]^2}{2}+\frac{\left[\nabla\left(\vec{\omega}(t)\cdot\vec{\Omega}(x,z,t)\right)\right]^2}{2}\cancel{+\nabla\left[\vec{\omega}(t)\cdot\vec{\Omega}(x,z,t)\right]\cdot\nabla\varphi(x,z,t)}dQ$$

$$-\rho\int_{Q(t)}\frac{\partial\varphi(x,z,t)}{\partial t}+\frac{[\nabla\varphi(x,z,t)]^2}{2}-\vec{g}\cdot\vec{r}(x,z)dQ$$

$$=-\rho\int_{Q(t)}x\ddot{\eta}_1(t)+z\ddot{\eta}_3(t)+\dot{\eta}_5(t)\cdot\frac{\partial\Omega_2(x,z,t)}{\partial t}+\Omega_2(x,z,t)\cdot\ddot{\eta}_5(t)-[z\dot{\eta}_5(t)\ddot{\eta}_1(t)-x\dot{\eta}_5(t)\ddot{\eta}_3(t)]$$

$$-\frac{(\dot{\eta}_1(t))^2+(\dot{\eta}_3(t))^2}{2}dQ+\frac{1}{2}[\dot{\eta}_5(t)]^2J_{22}^1-\rho\int_{Q(t)}\frac{\partial\varphi(x,z,t)}{\partial t}+\frac{[\nabla\varphi(x,z,t)]^2}{2}-\vec{g}\cdot\vec{r}(x,z)dQ$$

$$=-\ddot{\eta}_1(t)\cdot\rho\int_{Q(t)}xdQ-\ddot{\eta}_3(t)\cdot\rho\int_{Q(t)}zdQ-\dot{\eta}_5(t)\cdot\rho\int_{Q(t)}\frac{\partial\Omega_2(x,z,t)}{\partial t}dQ-\ddot{\eta}_5(t)\cdot\rho\int_{Q(t)}\Omega_2(x,z,t)dQ$$

$$-\dot{\eta}_5(t)\ddot{\eta}_3(t)\cdot\rho\int_{Q(t)}xdQ+\dot{\eta}_5(t)\ddot{\eta}_1(t)\cdot\rho\int_{Q(t)}zdQ+\frac{(\dot{\eta}_1(t))^2+(\dot{\eta}_3(t))^2}{2}\cdot\rho\int_{Q(t)}dQ+\frac{1}{2}[\dot{\eta}_5(t)]^2J_{22}^1$$

$$-\rho\int_{Q(t)}\sum_{n=1}^{\infty}\frac{\mathrm{d}R_n(t)}{\mathrm{d}t}\varphi_n(x,z)+\frac{1}{2}\sum_{n,k=1}^{\infty}R_nR_k\left[\nabla\varphi_n(x,z)\cdot\nabla\varphi_k(x,z)\right]-xg_1-zg_3dQ$$

5.

$$\int_{Q(t)} \nabla \left[\vec{\omega}(t) \cdot \vec{\Omega}(x,z,t)\right] \cdot \nabla\varphi(x,z,t) - \vec{\omega}(t) \times \vec{r}(x,z) \cdot \nabla\varphi(x,z,t) dQ$$
$$= \int_{S_{Q(t)}} \nabla \left[\vec{\omega}(t) \cdot \vec{\Omega}(x,z,t)\right] \cdot \vec{n}\varphi(x,z,t) - \vec{\omega}(t) \times \vec{r}(x,z) \cdot \vec{n}\varphi(x,z,t) dS$$
$$= \int_{S_{Q(t)}} \vec{\omega}(t) \cdot \left[\frac{\partial \vec{\Omega}(x,z,t)}{\partial n} - \vec{r}(x,z) \times \vec{n}\right] \varphi(x,z,t) dS = 0$$

6.

$$-\rho \int_{Q(t)} \frac{1}{2}\nabla \left[\vec{\omega}(t) \cdot \vec{\Omega}(x,z,t)\right] \cdot \nabla \left[\vec{\omega}(t) \cdot \vec{\Omega}(x,z,t)\right] - \vec{\omega}(t) \times \vec{r}(x,z) \cdot \nabla \left[\vec{\omega}(t) \cdot \vec{\Omega}(x,z,t)\right] dQ$$
$$\overset{\text{Gauss theorem}}{=\!=\!=} -\rho \int_{S(t)+\Sigma(t)} \frac{1}{2}\nabla \left[\vec{\omega}(t) \cdot \vec{\Omega}(x,z,t)\right] \cdot \vec{n} \left[\vec{\omega}(t) \cdot \vec{\Omega}(x,z,t)\right] - \vec{\omega}(t) \times \vec{r}(x,z) \cdot \vec{n} \left[\vec{\omega}(t) \cdot \vec{\Omega}(x,z,t)\right] dS$$
$$= -\rho \int_{S(t)+\Sigma(t)} \frac{1}{2}\vec{\omega}(t) \cdot \frac{\partial \vec{\Omega}(x,z,t)}{\partial n} \cdot \left[\vec{\omega}(t) \cdot \vec{\Omega}(x,z,t)\right] - \vec{\omega}(t) \cdot \frac{\partial \vec{\Omega}(x,z,t)}{\partial n} \cdot \left[\vec{\omega}(t) \cdot \vec{\Omega}(x,z,t)\right] dS$$
$$= \frac{1}{2}\rho \int_{S(t)+\Sigma(t)} \left[\vec{\omega}(t) \cdot \vec{\omega}(t)\right] \cdot \left[\vec{\Omega}(x,z,t) \cdot \frac{\partial \vec{\Omega}(x,z,t)}{\partial n}\right] dS$$
$$= \frac{1}{2}\rho \int_{S(t)+\Sigma(t)} \left[\dot{\eta}_5(t)\right]^2 \cdot \left[\Omega_2(x,z,t) \cdot \frac{\partial \Omega_2(x,z,t)}{\partial n}\right] dS$$
$$= \frac{1}{2} \left[\dot{\eta}_5(t)\right]^2 J^1_{22}$$

7. **Inertia tensor component**

$$J^1_{22}(x,z,t) = \rho \int_{S(t)+\Sigma(t)} \Omega_2(x,z,t) \frac{\partial \Omega_2(x,z,t)}{\partial n} dS \qquad (40)$$

5 Nonlinear Asymptotic Theories for a 2D Rectangular tank

5.1 Second order differential equation with constant coefficients

Second order differential equation with constant coefficients:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + k_1 \frac{\mathrm{d}y}{\mathrm{d}x} + k_2 y = f(x)$$
$$\lambda^2 + k_1\lambda + k_2 = 0$$

Table 1: General solution of homogeneous second order differential equation with constant coefficients.

	λ_1, λ_2	y_c
1	$\lambda_1 \neq \lambda_2 \geq 0$	$y_c = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
2	$\lambda_1 = \lambda_2 = \lambda \geq 0$	$y_c = (C_1 + C_2 x) e^{\lambda x}$
3	$\lambda = \alpha \pm \beta i$	$y_c = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

Table 2: Particular solution of non-homogeneous second order differential equation with constant coefficients.

	$f(x)$	y_p
1	be^{ax}	Ae^{ax}
2	$ax^n + \ldots$	$C_n x^n + C_{n-1} x^{n-1} + \ldots + C_0$
3	$P \cos ax, Q \sin ax,$ or $P \cos ax + Q \sin ax$	$A \cos ax + B \sin ax$

5.2 Steady state resonant solutions and their stability for a Duffing-like mechanical system

Nonlinear Duffing oscillator:

$$\ddot{\beta} + \omega_0^2 \left(\beta + K \beta^3 \right) = -\frac{\eta_a}{L} \omega^2 \cos(\omega t)$$

6 Mathieu's Equations

6.1 Floquet theory

How to define strength of nonlinearity of odes and pdes?
Perturbation and asymptotic approximation are only valid for weakly nonlinear odes and pdes.

Floquet theory

$$
\begin{aligned}
&\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x, \quad A(t+T) = A(t) \\
&\Longrightarrow \\
&y_i(t) = \lambda_i^{t/T} p_i(t), \quad p_i(t+T) = p_i(t), \\
&\lambda_i \text{ eigenvalues of } C = X(0)^{-1}X(T) = X(T).
\end{aligned} \tag{41}
$$

1. If any of one $|\lambda_i| > 1$, the solution of Mathieu's equation will be unstable (one unbounded solution exist).
2. If every $|\lambda_i| < 1$, the solution of Mathieu's equation will be stable (all solutions bounded).
3. If every $|\lambda_i| = 1$, the solution of Mathieu's equation will be period of T or $2T$.
4. $X(t) = [x_1(t), x_2(t), \dots, x_n(t)]$ period is T or $2T \Longrightarrow$
   $x_1 = x$ period is T or $2T \Longrightarrow x = \sum_{i=1}^{\infty} \cdots$

1. $x_1(t)_{n\times 1}, x_2(t)_{n\times 1}, \dots, x_n(t)_{n\times 1}$ linearly independent solution
2. $X(t)_{n\times n} = [x_1, x_2, \dots, x_n]$ non-singular, *fundamental solution*.
3. $X(t+T)$ fundamental solution.
4. Choose $X(0) = I$.
   Why can we manually choose $X(0) = I$? $\Longrightarrow$
   $Y(t) = X(t)C$ is a fundamental solution. Choose $C = X(0)^{-1}$ (non-singular) to make $Y(0) = X(0)X(0)^{-1} = I$.
5. $Y(t) = X(t)C$ fundamental solution.
6. Key function: $X(t+T) = X(t)C$, $C = X(0)^{-1}X(T) = X(T)$
7. $C = X^{-1}(t)X(t+T) \overset{C\text{: time-independent}}{=} X^{-1}(0)X(T) = X(T)$
8. $X(0+nT) = C^n \Longrightarrow$ iterates of a Poincare map corresponding to the surface of section $\Sigma : t = 0$ (mode2π). The question of the boundness of solutions is intimately connected to the matrix C.
9. To solve $X(t+T) = X(t)C \Longrightarrow$ Transform $X(t)$ to normal coordinates:

(a) $Y(t) = X(t)V$ fundamental solution

$$V^{-1}CV = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

(b) $Y(t+T) = X(t+T)V = X(t)CV = Y(t)V^{-1}CV = Y(t)\Lambda \overset{\text{See this.}}{\Longrightarrow}$

$$\begin{aligned}
&[y_1(t+T), y_2(t+T), \ldots, y_n(t+T)] \\
&= [y_1(t), y_2(t), \ldots, y_n(t)] \cdot \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\
&= [\lambda_1 y_1(t), \lambda_2 y_2(t), \ldots, \lambda_n y_n(t)]
\end{aligned}$$

(c) $y_i(t+T) = \lambda_i y_i(t) \Longrightarrow y_i(t) = \lambda_i^{kt} p_i(t)$

$$\begin{aligned}
y_i(t+T) &= \lambda_i^{k(t+T)} p_i(t+T) \\
&= \lambda_i y_i(t) = \lambda_i(\lambda_i^{kt} p_i(t)) = \lambda_i^{kt+1} p_i(t) \Longrightarrow \\
k &= \frac{1}{T}, \quad p_i(t+T) = p_i(t) \Longrightarrow
\end{aligned}$$

$$y_i(t) = \lambda_i^{t/T} p_i(t), \quad p_i(t+T) = p_i(t), \quad \lambda_i \text{ eigenvalues of } C = X(T)$$

10. If any of one $|\lambda_i| > 1$, the solution of Mathieu's equation will be unstable (one unbounded solution exist).
If every $|\lambda_i| < 1$, the solution of Mathieu's equation will be stable (all solutions bounded).
If every $|\lambda_i| = 1$, the solution of Mathieu's equation will be period of T or $2T$.

11. $X(t) = [x_1(t), x_2(t), \ldots, x_n(t)]$ period is T or $2T \Longrightarrow$
$x_1 = x$ period is T or $2T \Longrightarrow x = \sum_{i=1}^{\infty} \cdots$

6.2 Hill's equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + f(t)x = 0, \quad f(t+T) = f(t) \tag{42}$$

1. $x_1 = x, x_2 = \frac{\mathrm{d}x}{\mathrm{d}t} \Rightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -f(t) & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2. $X(t)=\begin{bmatrix}x_{11}(t) & x_{21}(t)\\ x_{12}(t) & x_{22}(t)\end{bmatrix}$, $X(0)=\begin{bmatrix}x_{11}(0) & x_{21}(0)\\ x_{12}(0) & x_{22}(0)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

3. $C=X(T)=\begin{bmatrix}x_{11}(T) & x_{21}(T)\\ x_{12}(T) & x_{22}(T)\end{bmatrix}$

4. λ_i of C $\Longrightarrow$

$$\begin{aligned}&\begin{vmatrix}\lambda-x_{11}(T) & -x_{21}(T)\\ -x_{12}(T) & \lambda-x_{22}(T)\end{vmatrix}\\ &=\lambda^2-\left[x_{11}(T)+x_{22}(T)\right]\lambda+\left[x_{11}(T)x_{22}(T)-x_{12}(T)x_{21}(T)\right]\\ &=\lambda^2-\left[\text{tr}(C)\right]\lambda+\underbrace{\det(C)}_{=1=\lambda_1\lambda_2}=0\\ &\Longrightarrow\end{aligned}$$

$$\lambda_{1,2}=\frac{\text{tr}C\pm\sqrt{(\text{tr}C)^2-4}}{2},\quad \lambda_1\lambda_2=\det(C)=1$$

Define $W(t)=\det C=x_{11}(t)x_{22}(t)-x_{12}(t)x_{21}(t)$

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}t}&=x_{11}'(t)x_{22}(t)+x_{11}(t)x_{22}'(t)-x_{12}'(t)x_{21}(t)-x_{12}(t)x_{21}'(t)\\ &=x_{12}(t)x_{22}(t)-f(t)x_{11}(t)x_{21}(t)+f(t)x_{11}(t)x_{21}(t)-x_{12}(t)x_{22}(t)\\ &=0\\ &\Longrightarrow W(t)=\text{const}=W(0)=x_{11}(0)x_{22}(0)-x_{12}(0)x_{21}(0)=1\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x_{11}\\ x_{12}\end{bmatrix}&=\begin{bmatrix}0 & 1\\ -f(t) & 0\end{bmatrix}\begin{bmatrix}x_{11}\\ x_{12}\end{bmatrix}\Longrightarrow x_{11}'=x_{12}, x_{12}'=-f(t)x_{11}\\ \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}x_{21}\\ x_{22}\end{bmatrix}&=\begin{bmatrix}0 & 1\\ -f(t) & 0\end{bmatrix}\begin{bmatrix}x_{21}\\ x_{22}\end{bmatrix}\Longrightarrow x_{21}'=x_{22}, x_{22}'=-f(t)x_{21}\end{aligned}$$

5. $y_i(t)=\lambda_i^{t/T}p_i(t)$, $p_i(t+T)=p_i(t)$,

   $\lambda_{1,2}=\frac{\text{tr}C\pm\sqrt{(\text{tr}C)^2-4}}{2}$ and $\lambda_1\lambda_2=\det(C)=1\Longrightarrow$

   (a) $|\text{tr}(C)|>2\Longrightarrow$ real toots, if $\lambda_1<1,\Longrightarrow\lambda_2=\frac{1}{\lambda_1}>1\Longrightarrow$ USTABLE, exponential growth in time.

   (b) $|\text{tr}(C)|<2\Longrightarrow$ complex conjugate toots, $\lambda_1=a+ib,\lambda_2=a-ib,\lambda_1\lambda_2=a^2+b^2=1\Longrightarrow|\lambda|=\sqrt{a^2+b^2}=1\Longrightarrow$ STABLE.

   (c) transition from stable to unstable: $|\text{tr}(C)=2|\Rightarrow$
   if $\text{tr}(C)=2\Longrightarrow\lambda_{1,2}=1,1\Longrightarrow Y(t),X(t)$ period solution with T;
   if $\text{tr}(C)=-2\Longrightarrow\lambda_{1,2}=-1,-1\Longrightarrow Y(t),X(t)$ period solution with $2T$.
   $\Longrightarrow$ on transition curves, $x(t)$ period with T or $2T$.

6.3 Harmonic balance

Apply Floquet theory to Mathieu's equation $\eta_3 = A_v \cos(\omega_v t) \Longrightarrow$

$$\frac{\mathrm{d}^2\beta(t)}{\mathrm{d}t^2} + \omega^2[1 - \epsilon \cos(\omega_v t)]\beta(t) = 0, \quad \epsilon = \frac{A_v \omega_v^2}{g} \tag{43}$$

1. $T = \frac{2\pi}{\omega_v}$, on the transition curves exist solutions of period $2T$ or $T \Longrightarrow$

$$\beta(t) = \sum_{n=0}^{\infty} \left(a_n \cos \frac{n\omega_v t}{2} + b_n \sin \frac{n\omega_v t}{2} \right) \Longrightarrow \beta(t) \text{ period } 2T \tag{44}$$

   When $a_{\mathrm{odd}} = b_{\mathrm{odd}} = 0 \Longrightarrow x(t)$ period T.

2. Substitute (44) into Mathieu equation (43)

$$\begin{cases} \displaystyle\sum_{n=0}^{\infty} \left[a_n \left(1 - \frac{n^2}{4\Omega^2}\right) \cos\!\left(\frac{n}{2}\omega_v t\right)\right] \\ \displaystyle -\frac{\epsilon}{2} \sum_{n=0}^{\infty} a_n \left[\cos\left(\omega_v t \left(\frac{n}{2}+1\right)\right) + \cos\left(\omega_v t \left(\frac{n}{2}-1\right)\right)\right] = 0 \\ \displaystyle\sum_{n=0}^{\infty} \left[b_n \left(1 - \frac{n^2}{4\Omega^2}\right) \sin\!\left(\frac{n}{2}\omega_v t\right)\right] \\ \displaystyle -\frac{\epsilon}{2} \sum_{n=0}^{\infty} b_n \left[\sin\left(\omega_v t \left(\frac{n}{2}+1\right)\right) + \sin\left(\omega_v t \left(\frac{n}{2}-1\right)\right)\right] = 0 \end{cases}$$

3. a_{even}:

$$\begin{aligned}
n = 0: &\quad a_0 \cos 0 - \epsilon a_0 \cos(\omega_v t) \\
n = 2: &\quad a_2 \left(1 - \frac{1}{\Omega^2}\right) \cos(\omega_v t) - \frac{\epsilon}{2} a_2 [\cos(2\omega_v t) + \cos 0] \\
n = 4: &\quad a_4 \left(1 - \frac{4}{\Omega^2}\right) \cos(2\omega_v t) - \frac{\epsilon}{2} a_4 [\cos(3\omega_v t) + \cos(\omega_v t)] \\
n = 6: &\quad a_6 \left(1 - \frac{9}{\Omega^2}\right) \cos(3\omega_v t) - \frac{\epsilon}{2} a_6 [\cos(4\omega_v t) + \cos(2\omega_v t)] \\
n = 8: &\quad a_8 \left(1 - \frac{16}{\Omega^2}\right) \cos(4\omega_v t) - \frac{\epsilon}{2} a_8 [\cos(5\omega_v t) + \cos(3\omega_v t)] \\
n = 2k: &\quad a_{2k} \left(1 - \frac{k^2}{\Omega^2}\right) \cos(k\omega_v t) - \frac{\epsilon}{2} a_{2k} [\cos((k+1)\omega_v t) + \cos((k-1)\omega_v t)]
\end{aligned}$$

$$\left[a_0 - \frac{\epsilon}{2}a_2\right]\cos 0 = 0$$
$$\left[-\epsilon a_0 + a_2\left(1 - \frac{1}{\Omega^2}\right) - \frac{\epsilon}{2}a_4\right]\cos(\omega_v t) = 0$$
$$\left[-\frac{\epsilon}{2}a_2 + a_4\left(1 - \frac{4}{\Omega^2}\right) - \frac{\epsilon}{2}a_6\right]\cos(2\omega_v t) = 0$$
$$\left[-\frac{\epsilon}{2}a_4 + a_6\left(1 - \frac{9}{\Omega^2}\right) - \frac{\epsilon}{2}a_8\right]\cos(3\omega_v t) = 0$$
$$\left[-\frac{\epsilon}{2}a_6 + a_8\left(1 - \frac{16}{\Omega^2}\right) - \frac{\epsilon}{2}a_{10}\right]\cos(4\omega_v t) = 0$$
$$\left[-\frac{\epsilon}{2}a_{2k-2} + a_{2k}\left(1 - \frac{k^2}{\Omega^2}\right) - \frac{\epsilon}{2}a_{2k+2}\right]\cos(k\omega_v t) = 0$$

$$\begin{vmatrix} a_0 & a_2 & a_4 & a_6 & a_8 & \cdots \\ 1 & -\frac{\epsilon}{2} & 0 & 0 & 0 & \cdots \\ -\epsilon & 1-\frac{1}{\Omega^2} & -\frac{\epsilon}{2} & 0 & 0 & \cdots \\ 0 & -\frac{\epsilon}{2} & 1-\frac{4}{\Omega^2} & -\frac{\epsilon}{2} & 0 & \cdots \\ 0 & 0 & -\frac{\epsilon}{2} & 1-\frac{9}{\Omega^2} & -\frac{\epsilon}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0$$

4. b_{even}:

$$n = 2: \quad b_2\left(1 - \frac{1}{\Omega^2}\right)\sin(\omega_v t) - \frac{\epsilon}{2}b_2\sin(2\omega_v t)$$
$$n = 4: \quad b_4\left(1 - \frac{4}{\Omega^2}\right)\sin(2\omega_v t) - \frac{\epsilon}{2}b_4[\sin(3\omega_v t) + \sin(\omega_v t)]$$
$$n = 6: \quad b_6\left(1 - \frac{9}{\Omega^2}\right)\sin(3\omega_v t) - \frac{\epsilon}{2}b_6[\sin(4\omega_v t) + \sin(2\omega_v t)]$$
$$n = 8: \quad b_8\left(1 - \frac{16}{\Omega^2}\right)\sin(4\omega_v t) - \frac{\epsilon}{2}b_8[\sin(5\omega_v t) + \sin(3\omega_v t)]$$
$$n = 2k: \quad b_{2k}\left(1 - \frac{k^2}{\Omega^2}\right)\sin(k\omega_v t) - \frac{\epsilon}{2}b_{2k}[\sin((k+1)\omega_v t) + \sin((k-1)\omega_v t)]$$

$$\left[b_2\left(1-\frac{1}{\Omega^2}\right)-\frac{\epsilon}{2}b_4\right]\sin(\omega_v t)=0$$
$$\left[-\frac{\epsilon}{2}b_2+b_4\left(1-\frac{4}{\Omega^2}\right)-\frac{\epsilon}{2}b_6\right]\sin(2\omega_v t)=0$$
$$\left[-\frac{\epsilon}{2}b_4+b_6\left(1-\frac{9}{\Omega^2}\right)-\frac{\epsilon}{2}b_8\right]\sin(3\omega_v t)=0$$
$$\left[-\frac{\epsilon}{2}b_6+b_8\left(1-\frac{16}{\Omega^2}\right)-\frac{\epsilon}{2}b_{10}\right]\sin(4\omega_v t)=0$$
$$\left[-\frac{\epsilon}{2}b_{2k-2}+b_{2k}\left(1-\frac{k^2}{\Omega^2}\right)-\frac{\epsilon}{2}b_{2k+2}\right]\sin(k\omega_v t)=0$$

$$\begin{vmatrix} b_2 & b_4 & b_6 & b_8 & \cdots \\ 1-\frac{1}{\Omega^2} & -\frac{\epsilon}{2} & 0 & 0 & \cdots \\ -\frac{\epsilon}{2} & 1-\frac{4}{\Omega^2} & -\frac{\epsilon}{2} & 0 & \cdots \\ 0 & -\frac{\epsilon}{2} & 1-\frac{9}{\Omega^2} & -\frac{\epsilon}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}=0$$

5. a_{odd}:

$$n=1:\quad a_1\left(1-\frac{1}{4\Omega^2}\right)\cos\left(\frac{1}{2}\omega_v t\right)-\frac{\epsilon}{2}a_1\left[\cos\left(\frac{3}{2}\omega_v t\right)+\cos\left(\frac{1}{2}\omega_v t\right)\right]$$
$$n=3:\quad a_3\left(1-\frac{9}{4\Omega^2}\right)\cos\left(\frac{3}{2}\omega_v t\right)-\frac{\epsilon}{2}a_3\left[\cos\left(\frac{5}{2}\omega_v t\right)+\cos\left(\frac{1}{2}\omega_v t\right)\right]$$
$$n=5:\quad a_5\left(1-\frac{25}{4\Omega^2}\right)\cos\left(\frac{5}{2}\omega_v t\right)-\frac{\epsilon}{2}a_5\left[\cos\left(\frac{7}{2}\omega_v t\right)+\cos\left(\frac{3}{2}\omega_v t\right)\right]$$
$$n=7:\quad a_7\left(1-\frac{49}{4\Omega^2}\right)\cos\left(\frac{7}{2}\omega_v t\right)-\frac{\epsilon}{2}a_7\left[\cos\left(\frac{9}{2}\omega_v t\right)+\cos\left(\frac{5}{2}\omega_v t\right)\right]$$
$$n=2k+1:\quad a_{2k+1}\left(1-\frac{(2k+1)^2}{4\Omega^2}\right)\cos\left(\frac{2k+1}{2}\omega_v t\right)$$
$$-\frac{\epsilon}{2}a_{2k+1}\left[\cos\left(\frac{2k+3}{2}\omega_v t\right)+\cos\left(\frac{2k-1}{2}\omega_v t\right)\right]$$

$$\left[\left(1-\frac{1}{4\Omega^2}-\frac{\epsilon}{2}\right)a_1-\frac{\epsilon}{2}a_3\right]\cos\left(\frac{1}{2}\omega_v t\right)=0$$
$$\left[-\frac{\epsilon}{2}a_1+\left(1-\frac{9}{4\Omega^2}\right)a_3-\frac{\epsilon}{2}a_5\right]\cos\left(\frac{3}{2}\omega_v t\right)=0$$
$$\left[-\frac{\epsilon}{2}a_3+\left(1-\frac{25}{4\Omega^2}\right)a_5-\frac{\epsilon}{2}a_7\right]\cos\left(\frac{5}{2}\omega_v t\right)=0$$
$$\left[-\frac{\epsilon}{2}a_5+\left(1-\frac{49}{4\Omega^2}\right)a_7-\frac{\epsilon}{2}a_9\right]\cos\left(\frac{7}{2}\omega_v t\right)=0$$
$$\left[-\frac{\epsilon}{2}a_{2k-1}+\left(1-\frac{(2k+1)^2}{4\Omega^2}\right)a_{2k+1}-\frac{\epsilon}{2}a_{2k+3}\right]\cos\left(\frac{2k+1}{2}\omega_v t\right)=0$$

$$\begin{vmatrix} a_1 & a_3 & a_5 & a_7 & a_9 & \cdots \\ 1-\frac{1}{4\Omega^2}-\frac{\epsilon}{2} & -\frac{\epsilon}{2} & 0 & 0 & 0 & \cdots \\ -\frac{\epsilon}{2} & 1-\frac{9}{4\Omega^2} & -\frac{\epsilon}{2} & 0 & 0 & \cdots \\ 0 & -\frac{\epsilon}{2} & 1-\frac{25}{4\Omega^2} & -\frac{\epsilon}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}=0$$

6. b_{odd}:

$$n=1: \quad b_1\left(1-\frac{1}{4\Omega^2}\right)\sin\left(\frac{1}{2}\omega_v t\right)-\frac{\epsilon}{2}b_1\left[\sin\left(\frac{3}{2}\omega_v t\right)-\sin\left(\frac{1}{2}\omega_v t\right)\right]$$
$$n=3: \quad b_3\left(1-\frac{9}{4\Omega^2}\right)\sin\left(\frac{3}{2}\omega_v t\right)-\frac{\epsilon}{2}b_3\left[\sin\left(\frac{5}{2}\omega_v t\right)+\sin\left(\frac{1}{2}\omega_v t\right)\right]$$
$$n=5: \quad b_5\left(1-\frac{25}{4\Omega^2}\right)\sin\left(\frac{5}{2}\omega_v t\right)-\frac{\epsilon}{2}b_5\left[\sin\left(\frac{7}{2}\omega_v t\right)+\sin\left(\frac{3}{2}\omega_v t\right)\right]$$
$$n=7: \quad b_7\left(1-\frac{49}{4\Omega^2}\right)\sin\left(\frac{7}{2}\omega_v t\right)-\frac{\epsilon}{2}b_7\left[\sin\left(\frac{9}{2}\omega_v t\right)+\sin\left(\frac{5}{2}\omega_v t\right)\right]$$
$$n=2k+1: \quad b_{2k+1}\left(1-\frac{(2k+1)^2}{4\Omega^2}\right)\sin\left(\frac{2k+1}{2}\omega_v t\right)$$
$$-\frac{\epsilon}{2}b_{2k+1}\left[\sin\left(\frac{2k+3}{2}\omega_v t\right)+\sin\left(\frac{2k-1}{2}\omega_v t\right)\right]$$

$$\left[\left(1-\frac{1}{4\Omega^2}+\frac{\epsilon}{2}\right)b_1-\frac{\epsilon}{2}b_3\right]\sin\left(\frac{1}{2}\omega_v t\right)=0$$
$$\left[-\frac{\epsilon}{2}b_1+\left(1-\frac{9}{4\Omega^2}\right)b_3-\frac{\epsilon}{2}b_5\right]\sin\left(\frac{3}{2}\omega_v t\right)=0$$
$$\left[-\frac{\epsilon}{2}b_3+\left(1-\frac{25}{4\Omega^2}\right)b_5-\frac{\epsilon}{2}b_7\right]\sin\left(\frac{5}{2}\omega_v t\right)=0$$
$$\left[-\frac{\epsilon}{2}b_5+\left(1-\frac{49}{4\Omega^2}\right)b_7-\frac{\epsilon}{2}b_9\right]\sin\left(\frac{7}{2}\omega_v t\right)=0$$
$$\left[-\frac{\epsilon}{2}b_{2k-1}+\left(1-\frac{(2k+1)^2}{4\Omega^2}\right)b_{2k+1}-\frac{\epsilon}{2}b_{2k+3}\right]\sin\left(\frac{2k+1}{2}\omega_v t\right)=0$$

$$\begin{vmatrix} b_1 & b_3 & b_5 & b_7 & b_9 & \cdots \\ 1-\frac{1}{4\Omega^2}+\frac{\epsilon}{2} & -\frac{\epsilon}{2} & 0 & 0 & 0 & \cdots \\ -\frac{\epsilon}{2} & 1-\frac{9}{4\Omega^2} & -\frac{\epsilon}{2} & 0 & 0 & \cdots \\ 0 & -\frac{\epsilon}{2} & 1-\frac{25}{4\Omega^2} & -\frac{\epsilon}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0$$

7 Computational Fluid Dynamics

RANS:

1. $\overline{u'} = 0$
2. $\overline{u'u'} \neq 0$
3. $\nabla^2 u^{'} = \frac{\partial}{\partial x}\left(\frac{\partial u^{'}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u^{'}}{\partial y}\right) = 0$
4. $$\overline{u^{'}\frac{\partial u^{'}}{\partial x}} = \frac{\partial \overline{u^{'}u^{'}}}{\partial x} - \overline{u^{'}\frac{\partial u^{'}}{\partial x}}$$

$$\overline{v^{'}\frac{\partial u^{'}}{\partial y}} = \frac{\partial \overline{u^{'}v^{'}}}{\partial y} - \overline{u^{'}\frac{\partial v^{'}}{\partial y}}$$

$$\begin{aligned} \overline{u^{'}\frac{\partial u^{'}}{\partial x}} + \overline{v^{'}\frac{\partial u^{'}}{\partial y}} &= \frac{\partial \overline{u^{'}u^{'}}}{\partial x} - \overline{u^{'}\frac{\partial u^{'}}{\partial x}} + \frac{\partial \overline{u^{'}v^{'}}}{\partial y} - \overline{u^{'}\frac{\partial v^{'}}{\partial y}} \\ &= \frac{\partial \overline{u^{'}u^{'}}}{\partial x} + \frac{\partial \overline{u^{'}v^{'}}}{\partial y} - \left[\overline{u^{'}\frac{\partial u^{'}}{\partial x}} + \overline{u^{'}\frac{\partial v^{'}}{\partial y}} = u^{'}\left(\frac{\partial u^{'}}{\partial x} + \frac{\partial v^{'}}{\partial y} = \nabla\cdot\overrightarrow{v'} = 0\right) = 0\right] \end{aligned}$$