Analysis

PENG

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Contents

1 Natural Number system

- This is how mathematics works - it treats its objects abstractly, caring only about what properties the objects have, not what the objects are or what they mean [1].
- It is pointless, at least from a mathematician's standpoint, as to argue about which model is the true one; as long as it obeys all the axioms and does all the right things, that's good enough to do maths.

1.1 The Peano Axioms

Axiom 1.1 (Peano axioms)**.**

1. *0 is a natural number.*
2. *If n is a natural number, then $n{+}{+}$ is also a natural number.*
3. $n{+}{+} \neq 0$
4. $n \neq m \Rightarrow n{+}{+} \neq m{+}{+}$

   $n{+}{+} = m{+}{+} \Rightarrow n = m$
5. *Principle of mathematical induction:*

   Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n{+}{+})$ is also true. Then $P(n)$ is true for every natural number n.

Definition 1.2.

$$\begin{aligned} 1 &:= 0{+}{+} \\ 2 &:= 1{+}{+} \\ 3 &:= 2{+}{+} \\ &\vdots \end{aligned}$$

1.2 Addition

Definition 1.3 (Addition)**.** *Let n and m be natural numbers.*

1. $0 + m := m$
2. $(n{+}{+}) + m := (n + m){+}{+}$

Proposition 1.4. *Let n and m be natural numbers. Then $n + m$ is also a natural number.*

Lemma 1.5. *Let n and m be natural numbers.*

1. $n + 0 = n$
2. $n + (m{+}{+}) = (n + m){+}{+}$

Proposition 1.6. *Let a, b, c be natural numbers.*

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$
3. $a + b = a + c \Rightarrow b = c$

Definition 1.7. *A natural number n is positive iff (if and only if) $n \neq 0$.*

Proposition 1.8. *Let a and b be natural numbers.*

1. *If a is positive, then $a + b$ is positive.*
2. $a + b = 0 \Rightarrow a = b = 0.$
3. *If a is positive, then there exists only one natural number b such that $b{+}{+} = a$.*

Proof. Prop. 1.8 (1).
$a + 0 = a$ is positive. Suppose that $a + b$ is positive. Then $a + b{+}{+} = (a + b){+}{+} \neq 0$ is positive. □

Proof. Prop. 1.8 (3).
Suppose that $a = b{+}{+}$. Then $a{+}{+} = (a){+}{+}$. □

Definition 1.9. *Let n and m be natural numbers.*

1. $n \geq m$ *or* $m \leq n$ *iff* $n = m + a$
2. $n > m$ *or* $m < n$ *iff* $n \geq m$ *and* $n \neq m$

Proposition 1.10. *Let a, b, c be natural numbers.*

1. $a \geq a$
2. *If $a \geq b$ and $b \geq c$, then $a \geq c$*
3. *If $a \geq b$ and $b \geq a$, then $a = b$*
4. $a \geq b$ *iff* $a + c \geq b + c$
5. $a < b$ *iff* $a{+}{+} \leq b$
6. $a < b$ *iff* $b = a + d$ *for some positive number d*

Proof. Prop. 1.10 (5).
$b = a + d$ and $b \neq a \Rightarrow d \neq 0 \Rightarrow d = n{+}{+}$
$b = a + (n{+}{+}) = (a{+}{+}) + n \Rightarrow a{+}{+} \leq b$ □

Proposition 1.11. *Let a and b be natural numbers. Then exactly one of following statement is true: $a < b$, $a = b$, or $a > b$.*

Proof. If $a < b$ and $a > b \overset{\text{Prop. 1.10 (3)}}{\Rightarrow} a = b$ contradiction □

Proposition 1.12 (Strong principle of induction)**.** *Let m_0 be a natural number. Suppose that for each $m \geq m_0$, if $P(m')$ is true for all $m_0 \leq m' < m$, then $P(m)$ is also true. Then $P(m)$ is true for all $m \geq m_0$.*

Proof. Let $Q(n)$ be the property that $P(m)$ is true for all $m_0 \leq m < n$.
Suppose that if $Q(n)$ is true, then $P(m)$ is also true.
Then $P(m)$ is true for all $m_0 \leq m < n{+}{+}$. □

Example 1.13. *Fibonacci numbers.*

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi} \tag{1.1}$$

$$F_{n+2} = F_{n+1} + F_n \tag{1.2}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ *and* $\psi = \frac{1-\sqrt{5}}{2}$

Exercise 1.14. *Principle of backwards induction.*
Let n be a natural number. Whenever $P(m{+}{+})$ is true, $P(m)$ is true. If $P(n)$ is true, then $P(m)$ is true for all $m \leq n$.

Proof. Suppose that if $P(n)$ is true, then $P(m)$ is true for all $m \leq n$. If $P(n{+}{+})$ is true, then $P(m)$ is true for all $m \leq n{+}{+}$. □

Exercise 1.15. *Principle of induction starting from the base case n.*
Let n be a natural number. Whenever $P(m)$ is true, $P(m{+}{+})$ is true. If $P(n)$ is true, then $P(m)$ is true for all $m \geq n$.

Proof. If $P(0)$ is true, suppose that $P(m)$ is true. Then $P(m{+}{+})$ is also true. Then $P(m)$ is true for all $m \geq 0$. Suppose that if $P(n)$ is true, then $P(m)$ is true for all $m \geq n$. If $P(n{+}{+})$ is true, then $P(m)$ is true for all $m \geq n{+}{+}$. □

1.3 Multiplication

Definition 1.16. *Let n and m be natural numbers.*

1. $0 \times m := 0$
2. $(n{+}{+}) \times m := (n \times m) + m$

Proposition 1.17. *Let n and m be natural numbers. Then $n \times m$ is also a natural number.*

Lemma 1.18. *Let n and m be natural numbers.*

1. $n \times 0 = 0 \times n$
2. $n \times (m{+}{+}) = (n \times m) + n$

Lemma 1.19 (Commutative)**.** *Let n and m be natural numbers.*

1. $n \times m = m \times n$
2. *$n \times m = 0$ iff at least one of n and m is equal to 0. If n and m are both positive, then nm is also positive.*

Proposition 1.20. *Let a, b, c be natural numbers.*

1. $a(b + c) = (b + c)a = bc + ca$
2. $(ab)c = a(bc)$
3. *If $a < b$ and c is positive, then $ac < nc$.*
4. *If $ac = bc$ and c is positive, then $a = b$.*

Lemma 1.21 (Euclid's division lemma)**.** *Let n be a natural number and q be a positive natural number. Then there exist natural numbers m and r, such that $0 \leq r < q$ and $n = mq + r$.*

Proof. $0 = 0 \times q + 0$ and $0 \leq r < q$. Suppose that $n = mq + r$ and $0 \leq r < q$. Then $n{+}{+} = mq + (r{+}{+})$ and $0 \leq r{+}{+} \leq q$. If $r{+}{+} = q$, $n{+}{+} = mq + q = (m+1)q + 0$. If $r{+}{+} < q$, $n{+}{+} = mq + r{+}{+}$. □

Definition 1.22 (Exponentiation)**.**

1. $m^0 := 1$ *and* $0^0 := 1$
2. $m^{n++} := m^n \times m$

2 Set Theory

Definition 2.1. *Set A is any unordered collection of objects. $x \in A$ if x lies in the collection, otherwise $x \notin A$.*

Axiom 2.2.

1. *If A is a set, then A is also an object.*
2. *$A = B$ iff every element of A belongs also to B and vice versa.*
3. *There exists an empty set $\emptyset$, such that every object $x \notin \emptyset$.*
4. *Let A be a non-empty set. Then there exist an object $x \in A$.*
5. *Singleton set $\{a\}$: $y \in \{a\}$ iff $y = a$. Pair set $\{a, b\}$: $y \in \{a, b\}$ iff $y = a$ or $y = b$.*
6. *$x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$.*
7. *$y \in \{x \in A : P(x) \text{ is true}\} \Leftrightarrow y \in A$ and $P(x)$ is true.*
8. *$z \in \{y : P(x, y) \text{ is true for some } x \in A\} \Leftrightarrow P(x, z)$ is true for some $x \in A$.*
9. *There exists a set $\mathbb{N}$, whose elements are called natural numbers, and an object $n{+}{+}$ assigned to every $n \in \mathbb{N}$, such that the Peano axioms 1.1 hold.*

Example 2.3. $\{a, a\} = \{a\}$

Exercise 2.4. *$\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}\}$ are all distinct.*

Proof. $y \notin \emptyset$.
$y = \emptyset \in \{\emptyset\}$.
$y = \{\emptyset\} \in \{\{\emptyset\}\}$.
$y = \emptyset$ or $y = \{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$. □

Lemma 2.5. *Let a, b are objects, and A, B, C are sets.*

1. $\{a, b\} = \{a\} \cup \{b\}$
2. $A \cup B = B \cup A$
3. $(A \cup B) \cup C = A \cup (B \cup C)$

Definition 2.6.

$$\{a, b, c\} := \{a\} \cup \{b\} \cup \{c\}$$
$$\{a, b, c, d\} := \{a\} \cup \{b\} \cup \{c\} \cup \{d\}$$
$$\vdots$$

Definition 2.7 (Subsets)**.**

1. *$A \subseteq B \Leftrightarrow$ for any object $x \in A$, then $x \in B$.*
2. *$A \subsetneq B$ if $A \subseteq B$ and $A \neq B$.*

Proposition 2.8. *Let A, B, C be sets.*

1. *If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

2. *If* $A \subseteq B$ *and* $B \subseteq A$*, then* $A = C$*.*

3. *If* $A \subsetneq B$ *and* $B \subsetneq C$*, then* $A \subsetneq C$*.*

Definition 2.9. *Let* A*,* B *be sets.*

1. $A \cap B := \{x \in A : x \in B\}$.

2. $A \backslash B := \{x \in A : x \notin B\}$.

Proposition 2.10. *Let* A*,* B*,* C *be sets, and* $\{A, B, C\} \subseteq X$*.*

1. $A \cup \emptyset = A$ *and* $A \cap \emptyset = \emptyset$*.*

2. $A \cup X = X$ *and* $A \cap X = A$*.*

3. $A \cup A = A$ *and* $A \cap A = A$*.*

4. $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$*.*

5. $(A \cup B) \cup C = A \cup (B \cup C)$ *and* $(A \cap B) \cap C = A \cap (B \cap C)$*.*

6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$*.*

7. $A \cup (X \backslash A) = X$ *and* $A \cap (X \backslash A) = \emptyset$*.*

8. *De Morgan laws:* $X \backslash (A \cup B) = (x \backslash A) \cap (X \backslash B)$ *and* $X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$*.*

Proof. Prop. 2.10 (5).
$x \in A \cap X \Leftrightarrow x \in A \Rightarrow A \cap X = A$. ☐

Proof. Prop. 2.10 (6).
$x \in A \cap (B \cup C) \Rightarrow x \in A$ and $(x \in B$ or $x \in C) \Rightarrow x \in A \cap B$ or $x \in A \cap C \Rightarrow x \in (A \cap B) \cup (A \cap C)$.
$x \in (A \cap B) \cup (A \cap C) \Rightarrow x \in (A \cap B)$ or $x \in (A \cap C) \Rightarrow x \in A$ and $x \in B \cup C \Rightarrow x \in A \cap (B \cup C)$. ☐

Lemma 2.11 (De mongan laws)**.**
!(A or B) = !A and !B.
!(A and B) = !A or !B.

Proof. Prop. 2.10 (8).
$x \in X \backslash (A \cup B) \Rightarrow x \in X$ and $(x \notin A$ and $x \notin B)$.
$x \in X \backslash (A \cap B) \Rightarrow x \in X$ and $(x \notin A$ or $x \notin B)$. ☐

Example 2.12.
$\{y : y = f(x) \textit{ for some } x \in A\} = \{f(x) : x \in A\}$.
$\{x \in A : P(x) \textit{ is true}\} \Rightarrow \{f(x) : x \in A; P(x) \textit{ is true}\}$.

Example 2.13. $\{n + 3 : n \in \mathbb{N}, 0 \leq n \leq 5\} = \{8 - m : m \in \mathbb{N}, 0 \leq m \leq 3\}$.

Exercise 2.14 (Axiom 2.2 (8) $\Rightarrow$ Axiom 2.2 (7))**.**
$y \in \{x : P(x) \textit{ is true for some } x \in A\} \Rightarrow y \in \{x \in A : P(x) \textit{ is true}\}$.

2.1 Functions

References

[1] Terence Tao. *Analysis I.* Vol. 1. Springer, 2022. ISBN: 978-981-19-7261-4.