

Analysis

PENG

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Contents

1	Introduction	1
2	Natural Number system	1
2.1	The Peano Axioms	1
2.2	Addition	2
2.3	Multiplication	3
3	Set Theory	4
3.1	Functions	5

1 Introduction

Mathematics treats its objects abstractly, caring only about what **properties** the objects have [TAO'2022' ANALYSIS']

2 Natural Number system

2.1 The Peano Axioms

Axiom 2.1. 0 is a natural number.

Axiom 2.2. If n is a natural number, then $n++$ is also a natural number.

Axiom 2.3. $n++ \neq 0$

Axiom 2.4.

- $n \neq m \Rightarrow n++ \neq m++$
- $n++ = m++ \Rightarrow n = m$

Axiom 2.5 (principle of mathematical induction). Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true for every natural number n .

Definition 2.6. $1 := 0++$, $2 := 1++$, $3 := 2++$, ...

Proposition 2.7. 3 is a natural number.

Proof. By Axiom 2.1, 0 is a natural number.

By Axiom 2.2, $0++ = 1$ is a natural number.

By Axiom 2.2, $1++ = 2$ is a natural number.

By Axiom 2.2, $2++ = 3$ is a natural number. □

2.2 Addition

Definition 2.8 (Addition). *Let n and m be natural numbers.*

1. $0 + m := m$
2. $(n++) + m := (n + m)++$

Proposition 2.9. *Let n and m be natural numbers. Then $n + m$ is also a natural number.*

Lemma 2.10. *Let n and m be natural numbers.*

1. $n + 0 = n$
2. $n + (m++) = (n + m)++$

Proposition 2.11. *Let a, b, c be natural numbers.*

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$
3. $a + b = a + c \Rightarrow b = c$

Definition 2.12. *A natural number n is positive iff (if and only if) $n \neq 0$.*

Proposition 2.13. *Let a and b be natural numbers.*

1. *If a is positive, then $a + b$ is positive.*
2. $a + b = 0 \Rightarrow a = b = 0$.
3. *If a is positive, then there exists only one natural number b such that $b++ = a$.*

Proof. Prop. 2.13 (1).

$a + 0 = a$ is positive. Suppose that $a + b$ is positive. Then $a + b++ = (a + b)++ \neq 0$ is positive. $\square$

Proof. Prop. 2.13 (3).

Suppose that $a = b++$. Then $a++ = (a)++$. $\square$

Definition 2.14. *Let n and m be natural numbers.*

1. $n \geq m$ or $m \leq n$ iff $n = m + a$
2. $n > m$ or $m < n$ iff $n \geq m$ and $n \neq m$

Proposition 2.15. *Let a, b, c be natural numbers.*

1. $a \geq a$
2. *If $a \geq b$ and $b \geq c$, then $a \geq c$*
3. *If $a \geq b$ and $b \geq a$, then $a = b$*
4. $a \geq b$ iff $a + c \geq b + c$
5. $a < b$ iff $a++ \leq b$
6. $a < b$ iff $b = a + d$ for some positive number d

Proof. Prop. 2.15 (5).

$b = a + d$ and $b \neq a \Rightarrow d \neq 0 \Rightarrow d = n++$

$b = a + (n++) = (a++) + n \Rightarrow a++ \leq b$ $\square$

Proposition 2.16. *Let a and b be natural numbers. Then exactly one of following statement is true: $a < b$, $a = b$, or $a > b$.*

Proof. If $a < b$ and $a > b \xrightarrow{\text{Prop. 2.15 (3)}} a = b$ contradiction $\square$

Proposition 2.17 (Strong principle of induction). *Let m_0 be a natural number. Suppose that for each $m \geq m_0$, if $P(m')$ is true for all $m_0 \leq m' < m$, then $P(m)$ is also true. Then $P(m)$ is true for all $m \geq m_0$.*

Proof. Let $Q(n)$ be the property that $P(m)$ is true for all $m_0 \leq m < n$.

Suppose that if $Q(n)$ is true, then $P(n)$ is also true.

Then $P(m)$ is true for all $m_0 \leq m < n++$. □

Example 2.18. *Fibonacci numbers.*

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi} \quad (2.1)$$

$$F_{n+2} = F_{n+1} + F_n \quad (2.2)$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$

Exercise 2.19. *Principle of backwards induction.*

Let n be a natural number. Whenever $P(m++)$ is true, $P(m)$ is true. If $P(n)$ is true, then $P(m)$ is true for all $m \leq n$.

Proof. Suppose that if $P(n)$ is true, then $P(m)$ is true for all $m \leq n$. If $P(n++)$ is true, then $P(m)$ is true for all $m \leq n++$. □

Exercise 2.20. *Principle of induction starting from the base case n .*

Let n be a natural number. Whenever $P(m)$ is true, $P(m++)$ is true. If $P(n)$ is true, then $P(m)$ is true for all $m \geq n$.

Proof. If $P(0)$ is true, suppose that $P(m)$ is true. Then $P(m++)$ is also true. Then $P(m)$ is true for all $m \geq 0$. Suppose that if $P(n)$ is true, then $P(m)$ is true for all $m \geq n$. If $P(n++)$ is true, then $P(m)$ is true for all $m \geq n++$. □

2.3 Multiplication

Definition 2.21. *Let n and m be natural numbers.*

1. $0 \times m := 0$
2. $(n++) \times m := (n \times m) + m$

Proposition 2.22. *Let n and m be natural numbers. Then $n \times m$ is also a natural number.*

Lemma 2.23. *Let n and m be natural numbers.*

1. $n \times 0 = 0 \times n$
2. $n \times (m++) = (n \times m) + n$

Lemma 2.24 (Commutative). *Let n and m be natural numbers.*

1. $n \times m = m \times n$
2. $n \times m = 0$ iff at least one of n and m is equal to 0. If n and m are both positive, then nm is also positive.

Proposition 2.25. *Let a, b, c be natural numbers.*

1. $a(b+c) = (b+c)a = bc+ca$
2. $(ab)c = a(bc)$
3. If $a < b$ and c is positive, then $ac < bc$.
4. If $ac = bc$ and c is positive, then $a = b$.

Lemma 2.26 (Euclid's division lemma). *Let n be a natural number and q be a positive natural number. Then there exist natural numbers m and r , such that $0 \leq r < q$ and $n = mq + r$.*

Proof. $0 = 0 \times q + 0$ and $0 \leq r < q$. Suppose that $n = mq + r$ and $0 \leq r < q$. Then $n++ = mq + (r++)$ and $0 \leq r++ \leq q$. If $r++ = q$, $n++ = mq + q = (m+1)q + 0$. If $r++ < q$, $n++ = mq + r++$. $\square$

Definition 2.27 (Exponentiation).

1. $m^0 := 1$ and $0^0 := 1$
2. $m^{n++} := m^n \times m$

3 Set Theory

Definition 3.1. Set A is any unordered collection of objects. $x \in A$ if x lies in the collection, otherwise $x \notin A$.

Axiom 3.2.

1. If A is a set, then A is also an object.
2. $A = B$ iff every element of A belongs also to B and vice versa.
3. There exists an empty set $\emptyset$, such that every object $x \notin \emptyset$.
4. Let A be a non-empty set. Then there exist an object $x \in A$.
5. Singleton set $\{a\}$: $y \in \{a\}$ iff $y = a$. Pair set $\{a, b\}$: $y \in \{a, b\}$ iff $y = a$ or $y = b$.
6. $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$.
7. $y \in \{x \in A : P(x) \text{ is true}\} \Leftrightarrow y \in A$ and $P(y)$ is true.
8. $z \in \{y : P(x, y) \text{ is true for some } x \in A\} \Leftrightarrow P(x, z) \text{ is true for some } x \in A$.
9. There exists a set $\mathbb{N}$, whose elements are called natural numbers, and an object $n++$ assigned to every $n \in \mathbb{N}$, such that the Peano axioms ?? hold.

Example 3.3. $\{a, a\} = \{a\}$

Exercise 3.4. $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$ are all distinct.

Proof. $y \notin \emptyset$.

$$y = \emptyset \in \{\emptyset\}.$$

$$y = \{\emptyset\} \in \{\{\emptyset\}\}.$$

$$y = \emptyset \text{ or } y = \{\emptyset\} \in \{\emptyset, \{\emptyset\}\}.$$

$\square$

Lemma 3.5. Let a, b are objects, and A, B, C are sets.

1. $\{a, b\} = \{a\} \cup \{b\}$
2. $A \cup B = B \cup A$
3. $(A \cup B) \cup C = A \cup (B \cup C)$

Definition 3.6.

$$\begin{aligned} \{a, b, c\} &:= \{a\} \cup \{b\} \cup \{c\} \\ \{a, b, c, d\} &:= \{a\} \cup \{b\} \cup \{c\} \cup \{d\} \\ &\vdots \end{aligned}$$

Definition 3.7 (Subsets). 1. $A \subseteq B \Leftrightarrow$ for any object $x \in A$, then $x \in B$.

2. $A \subsetneq B$ if $A \subseteq B$ and $A \neq B$.

Proposition 3.8. Let A, B, C be sets.

1. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

2. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

3. If $A \subsetneq B$ and $B \subsetneq C$, then $A \subsetneq C$.

Definition 3.9. Let A, B be sets.

1. $A \cap B := \{x \in A : x \in B\}$.

2. $A \setminus B := \{x \in A : x \notin B\}$.

Proposition 3.10. Let A, B, C be sets, and $\{A, B, C\} \subseteq X$.

1. $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

2. $A \cup X = X$ and $A \cap X = A$.

3. $A \cup A = A$ and $A \cap A = A$.

4. $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

5. $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.

6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

7. $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \emptyset$.

8. De Morgan laws: $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

Proof. Prop. 3.10 (5).

$$x \in A \cap X \Leftrightarrow x \in A \Rightarrow A \cap X = A. \quad \square$$

Proof. Prop. 3.10 (6).

$$x \in A \cap (B \cup C) \Rightarrow x \in A \text{ and } (x \in B \text{ or } x \in C) \Rightarrow x \in A \cap B \text{ or } x \in A \cap C \Rightarrow x \in (A \cap B) \cup (A \cap C).$$

$$x \in (A \cap B) \cup (A \cap C) \Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C) \Rightarrow x \in A \text{ and } x \in B \cup C \Rightarrow x \in A \cap (B \cup C). \quad \square$$

Lemma 3.11 (De Morgan laws).

$$\neg(A \text{ or } B) = \neg A \text{ and } \neg B.$$

$$\neg(A \text{ and } B) = \neg A \text{ or } \neg B.$$

Proof. Prop. 3.10 (8).

$$x \in X \setminus (A \cup B) \Rightarrow x \in X \text{ and } (x \notin A \text{ and } x \notin B).$$

$$x \in X \setminus (A \cap B) \Rightarrow x \in X \text{ and } (x \notin A \text{ or } x \notin B). \quad \square$$

Example 3.12.

$$\{y : y = f(x) \text{ for some } x \in A\} = \{f(x) : x \in A\}.$$

$$\{x \in A : P(x) \text{ is true}\} \Rightarrow \{f(x) : x \in A; P(x) \text{ is true}\}.$$

Example 3.13. $\{n + 3 : n \in \mathbb{N}, 0 \leq n \leq 5\} = \{8 - m : m \in \mathbb{N}, 0 \leq m \leq 3\}$.

Exercise 3.14 (Axiom 3.2 (8) $\Rightarrow$ Axiom 3.2 (7)).

$$y \in \{x : P(x) \text{ is true for some } x \in A\} \Rightarrow y \in \{x \in A : P(x) \text{ is true}\}.$$

3.1 Functions