Manifold Theory

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Contents

1 Set Theory

1.1 Relations

Definition 1.1. *Equivalence relation* $\sim$*:*

- *reflexive:* $x \sim x$
- *symmetric:* $x \sim y \Rightarrow y \sim x$
- *transitive:* $x \sim y, y \sim z \Rightarrow x \sim z$

Definition 1.2. *Equivalence class of equivalence relation* $\sim$*:*

$$[x] = \{y \in X : y \sim x\}$$

Example 1.1. Let $A = \{1, 2, 3\}$, $R = \{(x, y) : x = y\}$ on A

- reflexive: $x = y$
- symmetric: $x = y \Rightarrow y = x$
- transitive: $x = y, y = z \Rightarrow x = z$

Thus R is an equivalence relation.
Equivalence class:
$[1] = \{1\}$
$[2] = \{2\}$
$[3] = \{3\}$

Example 1.2. Let $A = \{1, 2, 3\}$, $R = \{(x, y) : x + y \text{ is even}\}$ on A

- reflexive: $x + x$ is even
- symmetric: $x + y$ is even $\Rightarrow y + x$ is even
- transitive: $x + y \text{ is even}, y + z \text{ is even} \Rightarrow \begin{cases} x \text{ is even}, y \text{ is even}, z \text{ is even} \\ x \text{ is odd}, y \text{ is odd}, z \text{ is odd} \end{cases} \Rightarrow x + z$ is even

Thus R is an equivalence relation.
Equivalence class:
$[1] = \{1, 3\}$
$[2] = \{2\}$
$[3] = \{1, 3\}$

2 Euclidean Spaces

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

Definition 2.1. *For vectors:*

$$x \cdot y = x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Definition 2.2. *Norm of x:*

$$|x| = \sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2}$$

Exercise 2.1.

$$\max\{|x_1|, |x_2|, \ldots, |x_n|\} \leq |x| \leq \sqrt{n} \max\{|x_1|, |x_2|, \ldots, |x_n|\}$$

Proof. Let $x_m = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$:

$$\sqrt{(x_m)^2} \leq \sqrt{(x_1)^2 + \cdots + (x_m)^2 + \cdots + (x_n)^2} \leq \sqrt{(x_m)^2 + \cdots + (x_m)^2}$$

□

Definition 2.3. *θ between x and y:*

3 Topological Spaces

Topology [1]

References

[1] J. M. Lee. *Introduction to Topological Manifolds*. Springer New York, 2010. ISBN: 978-1-4419-7939-1.