Manifold Theory

PENG

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Contents

1 Set Theory

1.1 Relations

Definition 1.1. *Equivalence relation* $\sim$*:*

- *reflexive:* $x \sim x$
- *symmetric:* $x \sim y \Rightarrow y \sim x$
- *transitive:* $x \sim y, y \sim z \Rightarrow x \sim z$

Definition 1.2. *Equivalence class of each* $x \in X$*:*
$[x] = \{y \in X : y \sim x\}$

Definition 1.3. *The collection of all* $[x]$ *is* $X/\sim$

Example 1.4. *Let* $A = \{1, 2, 3\}$*,* $R = \{(x, y) : x = y\}$ *on* A

- *reflexive:* $x = x$
- *symmetric:* $x = y \Rightarrow y = x$
- *transitive:* $x = y, y = z \Rightarrow x = z$

Thus R *is an equivalence relation.*
Equivalence class:
$[1] = \{1\}$
$[2] = \{2\}$
$[3] = \{3\}$

Example 1.5. *Let* $A = \{1, 2, 3\}$*,* $R = \{(x, y) : x + y \text{ is even}\}$ *on* A

- *reflexive:* $x + x$ *is even*
- *symmetric:* $x + y$ *is even* $\Rightarrow y + x$ *is even*
- *transitive:* $x + y \text{ is even}, y + z \text{ is even} \Rightarrow \begin{cases} x \text{ is even}, y \text{ is even}, z \text{ is even} \\ x \text{ is odd}, y \text{ is odd}, z \text{ is odd} \end{cases} \Rightarrow x + z \text{ is even}$

Thus R is an equivalence relation.
Equivalence class:
$[1] = \{1, 3\}$
$[2] = \{2\}$
$[3] = \{1, 3\}$

Definition 1.6. *A partition of X is a collection of disjoint nonempty subsets of X whose union is X.*

Example 1.7. $X = \{1, 2, 3\}$
$P = \{\{2\}, \{1, 3\}\}$
$P = \{\{1\}, \{3\}, \{2\}\}$
$P = \{\{1, 2, 3\}\}$
$\{\{1\}, \{2\}, \{\}\}$*: nonempty*
$\{\{1\}, \{2\}\}$*: union is not X*
$\{\{1, 3\}, \{2, 3\}\}$*: disjoint*
$\{\{1, 3\}, \{2, 4\}\}$*: union is not X*

Exercise 1.8. *Collection of equivalence class $\Rightarrow$ partition*
Partition $\Rightarrow$ unique equivalence relation

Proof. $X/\sim \Rightarrow$ partition:
For $\forall x_1, x_2 \in X$:
$[x_1] = \{y \in X : y \sim x_1\}$
$[x_2] = \{y \in X : y \sim x_2\}$
If $x_1 \nsim x_2, \forall y \in [x_1] \Rightarrow y \notin [x_2] \Rightarrow$ disjoint nonempty subsets
If $x_1 \sim x_2, \forall y \in [x_1] \Rightarrow y \in [x_2] \Rightarrow [x_1] = [x_2]$ □

Proof. Partition $\Rightarrow \sim$ □

2 Euclidean Spaces

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

Definition 2.1. *For vectors:*

$$x \cdot y = x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Definition 2.2. *Norm of x:*

$$|x| = \sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2}$$

Exercise 2.3.

$$\max\{|x_1|, |x_2|, \ldots, |x_n|\} \leq |x| \leq \sqrt{n} \max\{|x_1|, |x_2|, \ldots, |x_n|\}$$

Proof. Let $x_m = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$:

$$\sqrt{(x_m)^2} \leq \sqrt{(x_1)^2 + \cdots + (x_m)^2 + \cdots + (x_n)^2} \leq \sqrt{(x_m)^2 + \cdots + (x_m)^2}$$

□

Definition 2.4. *θ between x and y:*

3 Topological Spaces

Topology [1]
Four mechanisms (normal, segment, merge, torus) exist in every fluid point: fractal.

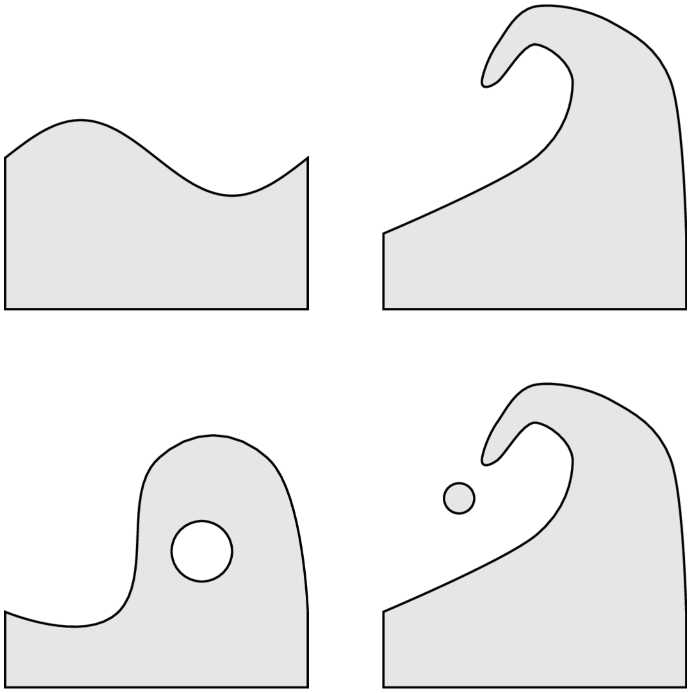

Figure 3.1: Sloshing topology.

References

[1] J.M. Lee. *Introduction to Topological Manifolds*. Springer New York, 2010. ISBN: 9781441979391.