Partial Differential Equations

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Contents

1 Four Fundamental Linear PDEs

Four fundamental linear PDEs [1]:

- Transport equation
- Laplace's equation
- Heat equation
- Wave equation

1.1 Transport Equation

$$u_t + c \cdot Du = 0 \tag{1.1}$$

where:

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, t \leq 0, Du = \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix}$$

Definition 1.1. *For vectors:*

$$x \cdot y = x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Example 1.1. For 1D and 2D cases:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + c_1\frac{\partial u}{\partial x} + c_2\frac{\partial u}{\partial y} = 0$$

Lemma 1.1.

$$\frac{dy}{d(x+C)} = \frac{dy}{dx}$$

Proof. Let $z := x + C$

$$\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{dy}{dx}$$

□

Example 1.2. Initial-value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + c \cdot \dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = g(x) \end{cases}$$

Let $z(s) := u(x + sc, t + s) \quad (s \in \mathbb{R})$

$$\frac{dz}{ds} = \frac{\partial u}{\partial (x+sc)} \frac{d(x+sc)}{ds} + \frac{\partial u}{\partial (t+s)} \frac{d(t+s)}{ds} = \frac{\partial u}{\partial x} c + \frac{\partial u}{\partial t} = 0$$

$$z(s) = u(x + sc, t + s) = C = u(x,t) = u(x + tc, 0) = g(x - tc)$$

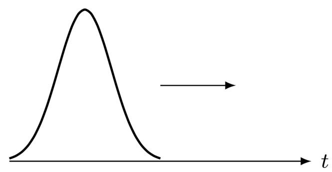

Figure 1.1: Transport equation.

Example 1.3. Nonhomogenous problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + c \cdot \dfrac{\partial u}{\partial x} = f(x,t) \\ u(x,0) = g(x) \end{cases}$$

Let $z(s) := u(x + sc, t + s)$

$$\frac{dz}{ds} = f(x + sc, t + s)$$

$$z(s) = u(x + sc, t + s) = \int f(x + sc, t + s) ds$$

$$z(0) - z(-t) = u(x,t) - u(x - tc, 0) = u(x,t) - g(x - tc) = \int_{-t}^{0} f(x + sc, t + s) ds$$

$$u(x,t) = g(x - tc) + \int_{-t}^{0} f(x + sc, t + s) ds$$

1.2 Laplace's Equation

$$\Delta u = 0 \tag{1.2}$$

Example 1.4. For 1D and 2D cases:

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Definition 1.2. *Outward derivative:*

$$\frac{\partial u}{\partial v} := \boldsymbol{v} \cdot Du$$

where $\boldsymbol{v}$ *is the unit outward normal vector of* ∂U*.*

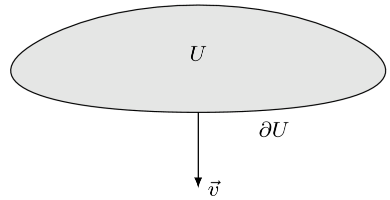

Figure 1.2: Outward derivative.

Example 1.5. For 1D and 2D cases:

$$\frac{\partial u}{\partial v} = v\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial v} = [v_1, v_2]\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{bmatrix} = v_1\frac{\partial u}{\partial x} + v_2\frac{\partial u}{\partial y}$$

Definition 1.3.

$$\nabla = \begin{bmatrix}\frac{\partial}{\partial x_1}\\ \frac{\partial}{\partial x_2}\\ \vdots\\ \frac{\partial}{\partial x_n}\end{bmatrix} \tag{1.3}$$

Theorem 1.2. *Gauss-Green theorem:*

$$\int_U \nabla \cdot \boldsymbol{u} dx = \int_{\partial U} \boldsymbol{u} \cdot \boldsymbol{v} dS \tag{1.4}$$

Example 1.6. For 1D and 2D cases:

$$\int_U \frac{\partial u}{\partial x} dx = \int_{\partial U} uv dS$$

$$\int_U \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} dx = \int_{\partial U} u_1 v_1 + u_2 v_2 dS$$

References

[1] Lawrence C. Evans. *Partial differential equations*. Vol. 19. American mathematical society, 2022. ISBN: 1-4704-6942-1.