Partial Differential Equations

PENG

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Contents

1 Four Fundamental Linear PDEs

Four fundamental linear PDEs [1]:

- Transport equation
- Laplace's equation
- Heat equation
- Wave equation

1.1 Transport Equation

Definition 1.1. *Gradient vector:*

$$Du = \nabla u := \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} \tag{1.1}$$

Definition 1.2. *Vector dot product ⇔ matrix multiplication:*

$$A_{n\times 1} \cdot B_{n\times 1} = A^T B = [a_1 a_2 \dots a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \tag{1.2}$$

Transport equation:

$$u_t + c \cdot Du = \frac{\partial u}{\partial t} + [c_1 c_2 \dots c_n] \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} = \frac{\partial u}{\partial t} + \sum_{i=n}^{n} c_i \frac{\partial u}{\partial x_i} = 0 \tag{1.3}$$

Example 1.3. *For 1D and 2D cases:*

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + c_1\frac{\partial u}{\partial x} + c_2\frac{\partial u}{\partial y} = 0$$

Lemma 1.4.

$$\frac{dy}{d(x+C)} = \frac{\frac{dy}{dx}}{\frac{d(x+C)}{dx}} = \frac{dy}{dx} \tag{1.4}$$

Example 1.5. *Initial-value problem:*

$$\begin{cases} \frac{\partial u}{\partial t} + c \cdot \frac{\partial u}{\partial x} = 0 \\ u(x,0) = g(x) \end{cases}$$

Let $z(s) := u(x+sc, t+s) \quad (s \in \mathbb{R})$

$$\frac{dz}{ds} = \frac{\partial u}{\partial(x+sc)} \frac{d(x+sc)}{ds} + \frac{\partial u}{\partial(t+s)} \frac{d(t+s)}{ds} = \frac{\partial u}{\partial x} c + \frac{\partial u}{\partial t} = 0$$

$$z(s) = u(x+sc, t+s) = C = u(x,t) = u(x-tc, 0) = g(x-tc)$$

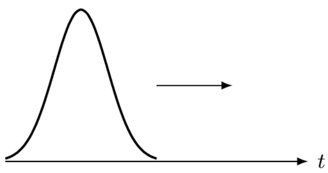

Figure 1.1: Transport equation.

Example 1.6. *Nonhomogenous problem:*

$$\begin{cases} \frac{\partial u}{\partial t} + c \cdot \frac{\partial u}{\partial x} = f(x,t) \\ u(x,0) = g(x) \end{cases}$$

Let $z(s) := u(x+sc, t+s)$

$$\frac{dz}{ds} = f(x+sc, t+s)$$

$$z(s) = u(x+sc, t+s) = \int f(x+sc, t+s) ds$$

$$z(0) - z(-t) = u(x,t) - u(x-tc, 0) = u(x,t) - g(x-tc) = \int_{-t}^{0} f(x+sc, t+s) ds$$

$$u(x,t) = g(x-tc) + \int_{-t}^{0} f(x+sc, t+s) ds$$

1.2 Laplace's Equation

Definition 1.7. *Laplace's equation:*

$$\Delta u = \nabla \cdot \nabla u = \nabla^2 u = \left[\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_n} \right] \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{1.5}$$

Example 1.8. *For 1D and 2D cases:*

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Definition 1.9. *Outward derivative:*

$$\frac{\partial u}{\partial v} := \boldsymbol{v} \cdot Du = [v_1 v_2 \dots v_n] \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} = \sum_{i=1}^{n} v_i \frac{\partial u}{\partial x_i}$$

where $\boldsymbol{v}$ *is the unit outer normal vector of* ∂U*.*

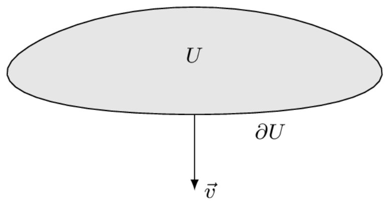

Figure 1.2: Outward derivative.

Example 1.10. *For 1D and 2D cases:*

$$\frac{\partial u}{\partial v} = v \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial v} = v_1 \frac{\partial u}{\partial x} + v_2 \frac{\partial u}{\partial y}$$

Theorem 1.11. *Gauss-Green theorem:*

$$\begin{aligned} \int_U \nabla \cdot \boldsymbol{u} dx &= \int_U \left[\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_n} \right] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} dx = \int_U \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} dx \\ &= \int_{\partial U} \boldsymbol{u} \cdot \boldsymbol{v} dS = \int_{\partial U} [u_1 u_2 \dots u_3] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} dS = \int_{\partial U} \sum_{i=1}^{n} u_i v_i dS \end{aligned} \tag{1.6}$$

Example 1.12. *For 1D and 2D cases:*

$$\int_U \frac{\partial u}{\partial x} dx = \int_{\partial U} uv dS$$

$$\int_U \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} dx = \int_{\partial U} u_1 v_1 + u_2 v_2 dS$$

Fundamental solution:

$$u(x) = v(r)$$

$$r = |x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$$

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

$$\frac{\partial u}{\partial x_i} = \frac{dv}{dr} \frac{x_i}{r}$$

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{d^2 v}{dr^2}\frac{x_i^2}{r^2} + \frac{dv}{dr}\frac{r^2 - x_i^2}{r^3}$$

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \frac{d^2 v}{dr^2} + \frac{dv}{dr}\frac{n-1}{r} = 0$$

For $n = 1$:

$$\frac{d^2 v}{dr^2} = 0$$

$$v(r) = r + C$$

For $n \geq 2$:

$$\frac{d^2 v}{dr^2} + \frac{dv}{dr}\frac{n-1}{r} = 0$$

$$\frac{v''(r)}{v'(r)} = \ln\left(|v'(r)|\right)' = \frac{1-n}{r}$$

$$\ln\left(|v'(r)|\right) = (1-n)\ln(r) + \ln(C) = \ln\left(\frac{C}{r^{n-1}}\right)$$

$$|v'(r)| = \frac{C}{r^{n-1}}$$

$$v'(r) = \frac{C}{r^{n-1}}$$

For $n = 2$:

$$v'(r) = \frac{C}{r}$$

$$v(r) = C_1 \ln(r) + C_2$$

For $n \geq 3$:

$$v(r) = \frac{C_1}{2-n}\frac{1}{r^{n-2}} + C_2 = \frac{C_1}{r^{n-2}} + C_2$$

Definition 1.13. *Fundamental solution of Laplace's equation:*

$$\Phi(|x|) := \begin{cases} -\frac{1}{2\pi}\ln|x| & (n=2) \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases} \tag{1.7}$$

Definition 1.14. *Poisson's equation:*

$$-\Delta u = -\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} = f \tag{1.8}$$

Theorem 1.15. *Solution of Poisson's equation:*

$$u(x) = \int_{y\in\mathbb{R}^n} \Phi(|x-y|)f(y)dy = \begin{cases} -\frac{1}{2\pi}\int_{\mathbb{R}^n}\ln\left(|x-y|\right)f(y)dy & (n=2) \\ \frac{1}{n(n-2)\alpha(n)}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2}}dy & (n \geq 3) \end{cases} \tag{1.9}$$

References

[1] Lawrence C Evans. *Partial differential equations*. Vol. 19. American mathematical society, 2022. ISBN: 1470469421.