

Partial Differential Equations

PENG

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Contents

1	Introduction	1
1.1	PDE	1
1.2	Solutions to PDE	3
2	Four Fundamental Linear PDEs	3
2.1	Transport Equation	3
2.2	Laplace's Equation	4
3	Asymptotics	6
3.1	Singular Perturbations	6

1 Introduction

1.1 PDE

Definition 1.1 (n -dimensional real Euclidean space). $\mathbb{R}^n$

Definition 1.2 (point in $\mathbb{R}^n$).

$$x := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad (1.1)$$

Definition 1.3 ($u(x)$).

$$u : U \rightarrow \mathbb{R} \Leftrightarrow u(x) := u(x_1, x_2, \dots, x_n) \quad (x \in U) \quad (1.2)$$

Definition 1.4 (standard coordinate vector).

$$e_i = i^{th} := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad (1.3)$$

Definition 1.5 (derivative).

$$u_{x_i} := \frac{\partial u}{\partial x_i}(x) = \lim_{h \rightarrow 0} \frac{u(x + he_i) - u(x)}{h} \quad (1.4)$$

Definition 1.6 (smooth). u is smooth provided u is *infinitely differentiable*.

Definition 1.7 (multi-index of order).

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \quad (1.5)$$

$$|\alpha| = \alpha_1 + \dots + \alpha_n \quad (1.6)$$

where $\alpha_i \geq 0$

Definition 1.8 (D^α).

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u \quad (1.7)$$

Definition 1.9 (D^k).

$$D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\} \quad (1.8)$$

Definition 1.10 (gradient vector).

$$Du = \nabla u := \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} \quad (1.9)$$

Definition 1.11 ($A : B$).

$$A : B = \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ij} \quad (1.10)$$

Definition 1.12 ($|A|$).

$$|A| = \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2} \quad (1.11)$$

Definition 1.13 (vector dot product and matrix multiplication).

$$A \cdot B = A^T B = [a_1 a_2 \dots a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^n a_i b_i \quad (1.12)$$

Definition 1.14 (radial derivative).

$$\begin{aligned} u_r &:= \frac{x}{|x|} \cdot Du = \frac{1}{|x|} [x_1 \dots x_n] \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} \\ &= \frac{\sum_{i=1}^n x_i \frac{\partial u}{\partial x_i}}{\sqrt{\sum_{i=1}^n x_i^2}} \quad (x \neq 0) \end{aligned} \quad (1.13)$$

Definition 1.15 (k^{th} -order PDE).

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0 \quad (1.14)$$

where $x \in U$, $u : U \rightarrow \mathbb{R}$, and $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times U \rightarrow \mathbb{R}$

Definition 1.16 (linear PDE).

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x) \quad (1.15)$$

$f \equiv 0 \Rightarrow \text{homogeneous}$

1.2 Solutions to PDE

Definition 1.17 (solutions to PDE [1]).

- *classic solutions*
 - *explicit, closed-form formula: polynomials, exp, cos, sin*
 - *or at least show such a solution exists, and to deduce various of its properties*
- *weak solutions and regularity (smoothness)*

Remark 1.18. *Existence is the prime attribute of what we study.*

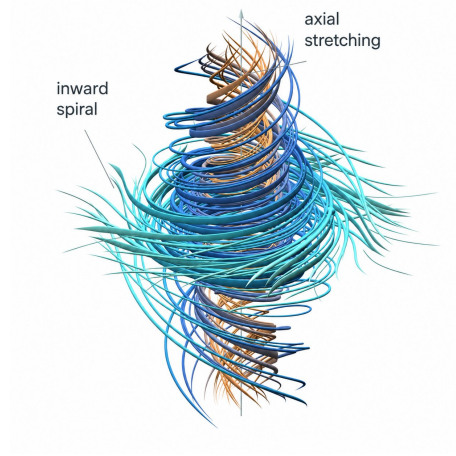


Figure 1.1: FINITE TIME BLOWUP FOR NAVIER-STOKES [2].

2 Four Fundamental Linear PDEs

Definition 2.1 (Fundamental linear PDEs).

- *Transport equation*
- *Laplace's equation*
- *Heat equation*
- *Wave equation*

Lemma 2.2. $\frac{dy}{d(x+b)} = \frac{dy}{dx}$

Proof.

$$\frac{dy}{d(x+b)} = \frac{\frac{dy}{dx}}{\frac{d(x+b)}{dx}} = \frac{dy}{dx}$$

□

2.1 Transport Equation

Definition 2.3 (Transport equation).

$$u_t + b \cdot Du = \frac{\partial u}{\partial t} + [b_1 b_2 \dots b_n] \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} = \frac{\partial u}{\partial t} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} = 0 \quad (2.1)$$

Example 2.4. For 1D and 2D cases:

$$\begin{aligned}\frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} &= 0 \\ \frac{\partial u}{\partial t} + b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} &= 0\end{aligned}$$

Theorem 2.5. For Eq. 2.1, u is constant on the line $(x, t) + (b, 1)s$ for $s \in \mathbb{R}$.

Proof. Let $z(s) := u(x + bs, t + s)$.

$$\begin{aligned}\frac{dz}{ds} &= \frac{\partial u}{\partial(t+s)} \frac{d(t+s)}{ds} + \frac{\partial u}{\partial(x+sb)} \frac{d(x+sb)}{ds} = \frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} \stackrel{\text{Eq. 2.1}}{=} 0 \\ z(s) &= u(x + sb, t + s) = C\end{aligned}$$

□

Proposition 2.6 (initial-value problem).

$$\begin{cases} \frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} = 0 \\ u(x, 0) = g(x) \end{cases}$$

$$u(x, t) \stackrel{\text{Theo. 2.5}}{=} u(x - tb, 0) = g(x - tb)$$

Proposition 2.7 (nonhomogenous problem).

$$\begin{cases} \frac{\partial u}{\partial t} + b \cdot \frac{\partial u}{\partial x} = f(x, t) \\ u(x, 0) = g(x) \end{cases}$$

Let $z(s) := u(x + sb, t + s)$.

$$\begin{aligned}\frac{dz}{ds} &= f(x + sb, t + s) \\ z(s) &= u(x + sb, t + s) = \int f(x + sb, t + s) ds \\ z(0) - z(-t) &= u(x, t) - u(x - tb, 0) = u(x, t) - g(x - tb) = \int_{-t}^0 f(x + sb, t + s) ds \\ u(x, t) &= g(x - tb) + \int_{-t}^0 f(x + sb, t + s) ds\end{aligned}$$

2.2 Laplace's Equation

Definition 2.8 (Laplace's equation).

$$\Delta u = \nabla \cdot \nabla u = \nabla^2 u = \left[\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_n} \right] \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = 0 \quad (2.2)$$

Example 2.9. For 1D and 2D cases:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= 0 \\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0\end{aligned}$$

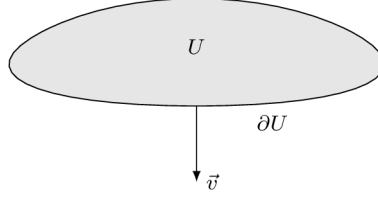


Figure 2.1: Outward derivative.

Definition 2.10 (outward derivative).

$$\frac{\partial u}{\partial v} := \mathbf{v} \cdot Du = [v_1 v_2 \dots v_n] \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} = \sum_{i=1}^n v_i \frac{\partial u}{\partial x_i}$$

where $\mathbf{v}$ is the unit outer normal vector of ∂U .

Example 2.11. For 1D and 2D cases:

$$\begin{aligned} \frac{\partial u}{\partial v} &= v \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial v} &= v_1 \frac{\partial u}{\partial x} + v_2 \frac{\partial u}{\partial y} \end{aligned}$$

Theorem 2.12 (Gauss-Green theorem).

$$\begin{aligned} \int_U \nabla \cdot \mathbf{u} dx &= \int_U \left[\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \dots \frac{\partial}{\partial x_n} \right] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} dx = \int_U \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} dx \\ &= \int_{\partial U} \mathbf{u} \cdot \mathbf{v} dS = \int_{\partial U} [u_1 u_2 \dots u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} dS = \int_{\partial U} \sum_{i=1}^n u_i v_i dS \end{aligned} \tag{2.3}$$

Example 2.13. For 1D and 2D cases:

$$\begin{aligned} \int_U \frac{\partial u}{\partial x} dx &= \int_{\partial U} u v dS \\ \int_U \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} dx &= \int_{\partial U} u_1 v_1 + u_2 v_2 dS \end{aligned}$$

Fundamental solution:

$$\begin{aligned} u(x) &= v(r) \\ r = |x| &= \sqrt{\sum_{i=1}^n x_i^2} \\ \frac{\partial r}{\partial x_i} &= \frac{x_i}{r} \\ \frac{\partial u}{\partial x_i} &= \frac{dv}{dr} \frac{x_i}{r} \\ \frac{\partial^2 u}{\partial x_i^2} &= \frac{d^2 v}{dr^2} \frac{x_i^2}{r^2} + \frac{dv}{dr} \frac{r^2 - x_i^2}{r^3} \\ \Delta u &= \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = \frac{d^2 v}{dr^2} + \frac{dv}{dr} \frac{n-1}{r} = 0 \end{aligned}$$

For $n = 1$:

$$\frac{d^2v}{dr^2} = 0$$

$$v(r) = r + C$$

For $n \geq 2$:

$$\frac{d^2v}{dr^2} + \frac{dv}{dr} \frac{n-1}{r} = 0$$

$$\frac{v''(r)}{v'(r)} = \ln(|v'(r)|)' = \frac{1-n}{r}$$

$$\ln(|v'(r)|) = (1-n) \ln(r) + \ln(C) = \ln\left(\frac{C}{r^{n-1}}\right)$$

$$|v'(r)| = \frac{C}{r^{n-1}}$$

$$v'(r) = \frac{C}{r^{n-1}}$$

For $n = 2$:

$$v'(r) = \frac{C}{r}$$

$$v(r) = C_1 \ln(r) + C_2$$

For $n \geq 3$:

$$v(r) = \frac{C_1}{2-n} \frac{1}{r^{n-2}} + C_2 = \frac{C_1}{r^{n-2}} + C_2$$

Definition 2.14 (fundamental solution of Laplace's equation).

$$\Phi(|x|) := \begin{cases} -\frac{1}{2\pi} \ln|x| & (n=2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{\frac{n-2}{2}}} & (n \geq 3) \end{cases} \quad (2.4)$$

Definition 2.15 (Poisson's equation).

$$-\Delta u = -\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = f \quad (2.5)$$

Theorem 2.16. *Solution of Poisson's equation:*

$$u(x) = \int_{y \in \mathbb{R}^n} \Phi(|x-y|) f(y) dy = \begin{cases} -\frac{1}{2\pi} \int_{\mathbb{R}^n} \ln(|x-y|) f(y) dy & (n=2) \\ \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{\frac{n-2}{2}}} dy & (n \geq 3) \end{cases} \quad (2.6)$$

3 Asymptotics

3.1 Singular Perturbations

References

- [1] Lawrence C Evans. *Partial differential equations*. Vol. 19. American mathematical society, 2022. ISBN: 1470469421.
- [2] OPENAI. "FINITE TIME BLOWUP FOR NAVIER-STOKES". In: (2026).