Partial Differential Equations

PENG

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Contents

1 Introduction

1.1 Notation

Definition 1.1 (n-dimensional real Euclidean space)**.** $\mathbb{R}^n$

Definition 1.2 (point in $\mathbb{R}^n$)**.**

$$x := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{1.1}$$

Definition 1.3 ($u(x)$)**.**

$$u : U \to \mathbb{R} \Leftrightarrow u(x) := u(x_1, \dots, x_n) \quad (x \in U) \tag{1.2}$$

Definition 1.4 (standard coordinate vector)**.**

$$e_i := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \tag{1.3}$$

Definition 1.5 (derivative)**.**

$$u_{x_i} := \frac{\partial u}{\partial x_i}(x) = \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h} \tag{1.4}$$

Definition 1.6 (smooth)**.** *u is smooth provided u is* ***infinitely differentiable****.*

Definition 1.7 (multi-index of order).

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \tag{1.5}$$

$$|\alpha| = \alpha_1 + \ldots + \alpha_n \tag{1.6}$$

where $\alpha_i \geq 0$

Definition 1.8 (D^α).

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} u \tag{1.7}$$

Definition 1.9 (D^k).

$$D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\} \tag{1.8}$$

Definition 1.10 (gradient vector).

$$Du = \nabla u := \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} \in \mathbb{R}^n \tag{1.9}$$

Definition 1.11 (Hessian matrix). *Real symmetrical matrices:*

$$D^2 u := \begin{bmatrix} u_{x_1 x_1} & \ldots & u_{x_1 x_n} \\ \vdots & \ddots & \vdots \\ u_{x_n x_1} & \ldots & u_{x_n x_n} \end{bmatrix} \tag{1.10}$$

Definition 1.12 ($A : B$).

$$A : B = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij} \tag{1.11}$$

Definition 1.13 ($|A|$).

$$|A| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2} \tag{1.12}$$

Definition 1.14 (vector dot product and matrix multiplication).

$$A \cdot B = A^T B = [a_1 \ldots a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_i b_i \tag{1.13}$$

Definition 1.15 (radial derivative).

$$\begin{aligned} u_r := & \frac{x}{|x|} \cdot Du = \frac{1}{|x|} [x_1 \ldots x_n] \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} \\ = & \frac{\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}}{\sqrt{\sum_{i=1}^{n} x_i^2}} \quad (x \neq 0) \end{aligned} \tag{1.14}$$

Definition 1.16 ($\boldsymbol{u}$).

$$\boldsymbol{u} : U \to \mathbb{R}^m \Leftrightarrow \boldsymbol{u} := \begin{bmatrix} u^1 \\ \vdots \\ u^m \end{bmatrix} \tag{1.15}$$

Definition 1.17 ($D^\alpha \boldsymbol{u}$)**.**

$$D^\alpha \boldsymbol{u} := \begin{bmatrix} D^\alpha u^1 \\ \vdots \\ D^\alpha u^n \end{bmatrix} \tag{1.16}$$

Definition 1.18 ($D^k \boldsymbol{u}$)**.**

$$D^k \boldsymbol{u} := \{D^\alpha \boldsymbol{u} : |\alpha| = k\} \tag{1.17}$$

Definition 1.19 (gradient matrix)**.**

$$D\boldsymbol{u} := \begin{bmatrix} u^1_{x_1} & \dots & u^1_{x_n} \\ \vdots & \ddots & \vdots \\ u^m_{x_1} & \dots & u^m_{x_n} \end{bmatrix} \tag{1.18}$$

1.2 PDE

Definition 1.20 (k^{th}-order PDE)**.**

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0 \tag{1.19}$$

where $x \in U$, $u : U \to \mathbb{R}$, *and* $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$

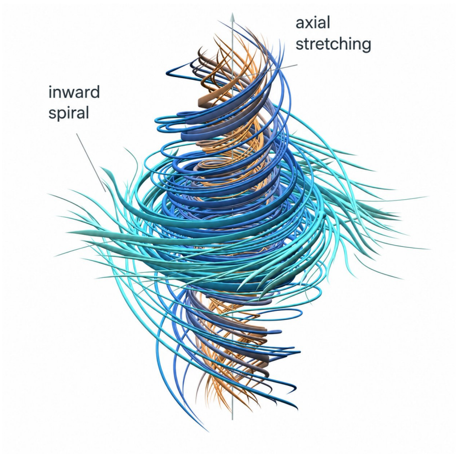

Figure 1.1: FINITE TIME BLOWUP FOR NAVIER–STOKES [2].

Definition 1.21 (linear PDE)**.**

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u = f(x) \tag{1.20}$$

$f \equiv 0 \Rightarrow$ *homogeneous*

Definition 1.22 (semi-linear)**.**

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \dots, Du, u, x) = 0 \tag{1.21}$$

Definition 1.23 (quasi-linear)**.**

$$\sum_{|\alpha| = k} a_\alpha(D^{k-1}u, \dots, Du, u, x) D^\alpha u + a_0(D^{k-1}u, \dots, Du, u, x) = 0 \tag{1.22}$$

Remark 1.24 (nonlinear)**.** *Full non-linearity depends on the highest order derivatives.*

Definition 1.25 (k^{th}-order system of PDE)**.**

$$F(D^k \boldsymbol{u}(x), D^{k-1}\boldsymbol{u}(x), \dots, D\boldsymbol{u}(x), \boldsymbol{u}(x), x) = 0 \tag{1.23}$$

where $x \in U$, $\boldsymbol{u} : U \to \mathbb{R}^m$, *and* $F : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \dots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times U \to \mathbb{R}^m$

1.3 Solutions to PDE

Definition 1.26 (solutions to PDE [1]).

- *classic solutions*
  - *explicit, closed-form formula: polynomials, exp, cos, sin*
  - *or at least show such a solution exists, and to deduce various of its properties*
- *weak solutions and regularity (smoothness)*

Remark 1.27. *Existence is the prime attribute of what we study.*

2 Four Fundamental Linear PDE

Definition 2.1 (Fundamental linear PDE).

- *Transport equation*
- *Laplace's equation*
- *Heat equation*
- *Wave equation*

Lemma 2.2. $\frac{dy}{d(x+b)} = \frac{dy}{dx}$

Proof.

$$\frac{dy}{d(x+b)} = \frac{\frac{dy}{dx}}{\frac{d(x+b)}{dx}} = \frac{dy}{dx}$$

□

2.1 Transport Equation

Definition 2.3 (Transport equation).

$$u_t + b \cdot Du = \frac{\partial u}{\partial t} + [b_1 b_2 \dots b_n] \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} = \frac{\partial u}{\partial t} + \sum_{i=n}^{n} b_i \frac{\partial u}{\partial x_i} = 0 \tag{2.1}$$

Example 2.4. *For 1D and 2D cases:*

$$\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = 0$$
$$\frac{\partial u}{\partial t} + b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} = 0$$

Theorem 2.5. *For Eq. 2.1, u is constant on the line $(x, t) + (b, 1)s$ for $s \in \mathbb{R}$.*

Proof. Let $z(s) := u(x + bs, t + s)$.

$$\frac{dz}{ds} = \frac{\partial u}{\partial(t+s)}\frac{d(t+s)}{ds} + \frac{\partial u}{\partial(x+sb)}\frac{d(x+sb)}{ds} = \frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} \overset{\text{Eq. 2.1}}{=} 0$$
$$z(s) = u(x + sb, t + s) = C$$

□

Proposition 2.6 (initial-value problem).

$$\begin{cases} \frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = 0 \\ u(x, 0) = g(x) \end{cases}$$

$$u(x, t) \overset{\text{Theo. 2.5}}{=} u(x - tb, 0) = g(x - tb)$$

Proposition 2.7 (nonhomogenous problem)**.**

$$\begin{cases} \frac{\partial u}{\partial t} + b \cdot \frac{\partial u}{\partial x} = f(x,t) \\ u(x,0) = g(x) \end{cases}$$

Let $z(s) := u(x+sb, t+s)$*.*

$$\frac{dz}{ds} = f(x+sb, t+s)$$
$$z(s) = u(x+sb, t+s) = \int f(x+sb, t+s) ds$$
$$z(0) - z(-t) = u(x,t) - u(x-tb, 0) = u(x,t) - g(x-tb) = \int_{-t}^{0} f(x+sb, t+s) ds$$
$$u(x,t) = g(x-tb) + \int_{-t}^{0} f(x+sb, t+s) ds$$

2.2 Laplace's Equation

Definition 2.8 (Laplace's equation)**.**

$$\Delta u = \nabla \cdot \nabla u = \nabla^2 u = \left[\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_n}\right] \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{2.2}$$

Example 2.9. *For 1D and 2D cases:*

$$\frac{\partial^2 u}{\partial x^2} = 0$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Definition 2.10 (outward derivative)**.**

$$\frac{\partial u}{\partial v} := \boldsymbol{v} \cdot Du = [v_1 v_2 \dots v_n] \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} = \sum_{i=1}^{n} v_i \frac{\partial u}{\partial x_i}$$

where $\boldsymbol{v}$ *is the unit outer normal vector of* ∂U*.*

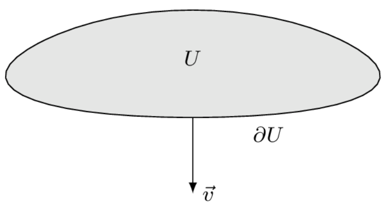

Figure 2.1: Outward derivative.

Example 2.11. *For 1D and 2D cases:*

$$\frac{\partial u}{\partial v} = v \frac{\partial u}{\partial x}$$
$$\frac{\partial u}{\partial v} = v_1 \frac{\partial u}{\partial x} + v_2 \frac{\partial u}{\partial y}$$

Theorem 2.12 (Gauss-Green theorem)**.**

$$\begin{aligned}\int_U \nabla \cdot \boldsymbol{u} dx = \int_U \left[\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n}\right]\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix} dx = \int_U \sum_{i=1}^{n}\frac{\partial u_i}{\partial x_i}dx \\ = \int_{\partial U} \boldsymbol{u}\cdot\boldsymbol{v} dS = \int_{\partial U}[u_1 u_2 \dots u_3]\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix} dS = \int_{\partial U}\sum_{i=1}^{n} u_i v_i dS\end{aligned} \tag{2.3}$$

Example 2.13. *For 1D and 2D cases:*

$$\int_U \frac{\partial u}{\partial x}dx = \int_{\partial U} uvdS$$
$$\int_U \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}dx = \int_{\partial U} u_1 v_1 + u_2 v_2 dS$$

Fundamental solution:

$$u(x) = v(r)$$
$$r = |x| = \sqrt{\sum_{i=1}^{n} x_i^2}$$
$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$
$$\frac{\partial u}{\partial x_i} = \frac{dv}{dr}\frac{x_i}{r}$$
$$\frac{\partial^2 u}{\partial x_i^2} = \frac{d^2 v}{dr^2}\frac{x_i^2}{r^2} + \frac{dv}{dr}\frac{r^2 - x_i^2}{r^3}$$
$$\Delta u = \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} = \frac{d^2 v}{dr^2} + \frac{dv}{dr}\frac{n-1}{r} = 0$$

For $n = 1$:

$$\frac{d^2 v}{dr^2} = 0$$

$$v(r) = r + C$$

For $n \geq 2$:

$$\frac{d^2 v}{dr^2} + \frac{dv}{dr}\frac{n-1}{r} = 0$$

$$\frac{v''(r)}{v'(r)} = \ln\left(|v'(r)|\right)' = \frac{1-n}{r}$$

$$\ln\left(|v'(r)|\right) = (1-n)\ln(r) + \ln(C) = \ln\left(\frac{C}{r^{n-1}}\right)$$

$$|v'(r)| = \frac{C}{r^{n-1}}$$

$$v'(r) = \frac{C}{r^{n-1}}$$

For $n = 2$:

$$v'(r) = \frac{C}{r}$$

$$v(r) = C_1 \ln(r) + C_2$$

For $n \geq 3$:

$$v(r) = \frac{C_1}{2-n} \frac{1}{r^{n-2}} + C_2 = \frac{C_1}{r^{n-2}} + C_2$$

Definition 2.14 (fundamental solution of Laplace's equation)**.**

$$\Phi(|x|) := \begin{cases} -\frac{1}{2\pi} \ln |x| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases} \tag{2.4}$$

Definition 2.15 (Poisson's equation)**.**

$$-\Delta u = -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = f \tag{2.5}$$

Theorem 2.16. *Solution of Poisson's equation:*

$$u(x) = \int_{y \in \mathbb{R}^n} \Phi(|x-y|) f(y) dy = \begin{cases} -\frac{1}{2\pi} \int_{\mathbb{R}^n} \ln(|x-y|) f(y) dy & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy & (n \geq 3) \end{cases} \tag{2.6}$$

3 Asymptotics

3.1 Singular Perturbations

References

[1] Lawrence C Evans. *Partial differential equations*. Vol. 19. American mathematical society, 2022. ISBN: 1470469421.

[2] OPENAI. "FINITE TIME BLOWUP FOR NAVIER–STOKES". In: (2026).